Classical Mechanics

Central force motion

Reduced mass

Kinetic energy of two objects 1 and 2: $K.E. = \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2$

However, it can be written in terms of two new variables

$\vec{R} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2}$ is the position of the center of mass

$\vec{r} = \vec{r}_1 - \vec{r}_2$ is the position of particle 1 with respect to particle 2.

Then the kinetic energy is

$$K.E. = \frac{1}{2}(m_1 + m_2)\dot{\vec{R}}^2 + \frac{1}{2}\mu\dot{\vec{r}}^2,$$

where the reduced mass μ is defined as $\mu = \frac{m_1 m_2}{m_1 + m_2}$.

<u>Example</u>: For the classical problem of the Sun and a planet the reduced mass is almost the same as the one of the planet (the maximum deviation would be for Jupiter, where the correction is $\approx 0.1\%$) so it is OK to assume the sun stationary and just treat the moving planet as a single object in a central force field.
On the other hand, if you treat classically the case of positronium, which is like hydrogen but with a positron instead of a proton, the correction is 50%!

Now, an additional simplification will happen if we take the origin of coordinates at the position of the center of mass, then the kinetic energy has the simple expression

$$K.E. = \frac{1}{2}\mu\dot{\vec{r}}^2$$

Kinetic energy in polar coordinates

If we change variables to $\dot{\vec{r}} = (r\cos\theta, r\sin\theta)$ it is straightforward to show that the kinetic energy can be written as the sum of two terms

$$K.E. = \frac{1}{2}\mu\dot{r}^2 + \frac{1}{2}\mu r^2\dot{\theta}^2$$

The first term is associated with the radial velocity and the second term with the tangential velocity.

Conservation of angular momentum

At this point consider that the force field is radial (it is a central force field), so it cannot apply torque on the system and so angular momentum is conserved. The angular momentum of the system is given by

$$\ell = \mu r^2 \dot{\theta},$$

but it is constant, so we can write the angular velocity in terms of the radius as follows $\dot{\theta} = \frac{\ell}{\mu r^2}$ and substituting in the kinetic energy we get

$$K.E. = \frac{1}{2}\mu \dot{r}^2 + \frac{1}{2}\mu r^2 \left(\frac{\ell}{\mu r^2}\right)^2 \rightarrow K.E. = \frac{1}{2}\mu \dot{r}^2 + \frac{1}{2}\frac{\ell^2}{\mu r^2}$$

Notice that the second term of the kinetic energy represents a diverging function as $r \rightarrow 0$, so it acts as a centrifugal barrier and in a sense behaves as a potential energy because it depends of r.

Total Energy

$$E = \frac{1}{2}\mu \dot{r}^2 + \frac{1}{2}\frac{\ell^2}{\mu r^2} + U(r)$$ where the last term is the potential energy.

Equation of motion

We solve for $\dot{r}$ in the equation above and find $\dot{r} = \pm\sqrt{\frac{2}{\mu}\left(E - \frac{1}{2}\frac{\ell^2}{\mu r^2} - U(r)\right)}$

This differential equation is separable and we can integrate both sides

$$\dot{r} = \frac{dr}{dt} \rightarrow t = \pm \int \frac{dr}{\sqrt{\frac{2}{\mu}\left(E - \frac{1}{2}\frac{\ell^2}{\mu r^2} - U(r)\right)}}$$

We can also obtain an equation for the trajectory. To do this notice that we can use the chain rule to write

$d\theta = \frac{d\theta}{dt}\frac{dt}{dr}dr = \frac{\dot{\theta}}{\dot{r}}dr$ and substituting this and $\dot{\theta} = \frac{\ell}{\mu r^2}$ in the equation for $\dot{r}$ we get

$$d\theta = \frac{\dot{\theta}}{\dot{r}}dr = \frac{\frac{\ell}{\mu r^2}}{\pm\sqrt{\frac{2}{\mu}\left(E - \frac{1}{2}\frac{\ell^2}{\mu r^2} - U(r)\right)}}dr$$ and in integral form

$$\theta = \pm\int \frac{\ell}{r^2}\frac{1}{\sqrt{2\mu\left(E-\frac{1}{2}\frac{\ell^2}{\mu r^2}-U(r)\right)}}dr$$

Planet around the sun

If we make $U(r) = -G\frac{m_{sun}m_{planet}}{r} = -\frac{k}{r}$ we have the very important case of planetary motion.

$$\theta = \pm\int \frac{\ell}{r^2}\frac{1}{\sqrt{2\mu\left(E-\frac{1}{2}\frac{\ell^2}{\mu r^2}+\frac{k}{r}\right)}}dr$$

To solve the integral we substitute $y = \frac{\ell}{r} \rightarrow r = \frac{\ell}{y}$

$$\theta = \mp\int \frac{1}{\sqrt{2\mu E + \left(\mu\frac{k}{\ell}\right)^2 - \left(y-\mu\frac{k}{\ell}\right)^2}}dy$$

Then we make another substitution $\left(y-\mu\frac{k}{\ell}\right) = \sqrt{2\mu E + \left(\mu\frac{k}{\ell}\right)^2}\cos\alpha$ we find that

$$\theta = \mp\cos^{-1}\left(\frac{y-\mu\frac{k}{\ell}}{\sqrt{2\mu E + \left(\mu\frac{k}{\ell}\right)^2}}\right)$$

$$\cos\theta = \mp\frac{y-\mu\frac{k}{\ell}}{\sqrt{2\mu E + \left(\mu\frac{k}{\ell}\right)^2}} \rightarrow y = \mu\frac{k}{\ell} \mp \sqrt{2\mu E + \left(\mu\frac{k}{\ell}\right)^2}\cos\theta$$

$$r = \frac{\ell}{\mu\frac{k}{\ell} \mp \sqrt{2\mu E + \left(\mu\frac{k}{\ell}\right)^2}\cos\theta} = \frac{\ell^2/(\mu k)}{1 \mp \sqrt{1+\frac{2E\ell^2}{\mu k^2}}\cos\theta}$$