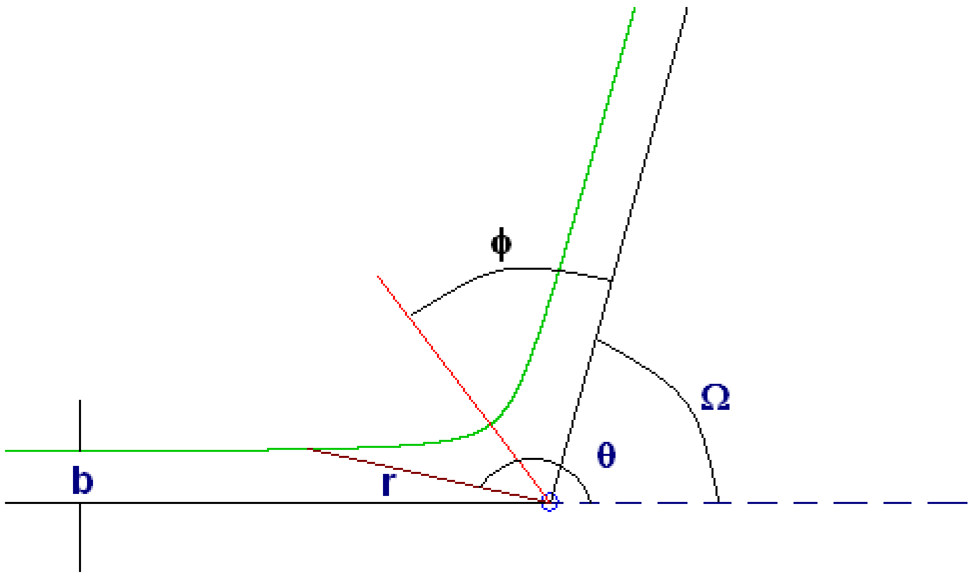
Classical Mechanics

Rutherford scattering

These notes are to demonstrate that the impact parameter in Rutherford scattering is

$$b = \frac{Z_1 Z_2 e^2}{8\pi\varepsilon_o E}\cot\left(\frac{\Omega}{2}\right)$$



The energy equation is

$$E = \frac{Z_1 Z_2 e^2}{4\pi\varepsilon_o r} + \frac{1}{2}mv^2,$$

However, the velocity can be written as two components: a tangential component plus a radial component:

$$E = \frac{Z_1 Z_2 e^2}{4\pi\varepsilon_o r} + \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right)$$

In this equation we can substitute the angular momentum: $L = mr^2\dot{\theta} \rightarrow \dot{\theta} = \frac{L}{mr^2}$, so:

$$E = \frac{Z_1 Z_2 e^2}{4\pi\varepsilon_o r} + \frac{1}{2}m\dot{r}^2 + \frac{1}{2}\frac{L^2}{mr^2}$$

Knowing that at closest approach $\dot{r} = 0$ we can use this equation to find $r_{minimum}$.

$$Er^2 - \frac{Z_1 Z_2 e^2}{4\pi\varepsilon_o} r - \frac{1}{2}\frac{L^2}{m} = 0 \rightarrow r_{\text{minimum}} = \frac{\frac{Z_1 Z_2 e^2}{4\pi\varepsilon_o} + \sqrt{\left(\frac{Z_1 Z_2 e^2}{4\pi\varepsilon_o}\right)^2 + 2E\frac{L^2}{m}}}{2E}$$

Also, notice that: $\dot{\theta} = \frac{L}{mr^2}$ can be written as: $\frac{d\theta}{dr}\dot{r} = \frac{L}{mr^2} \rightarrow d\theta = \frac{Ldr}{mr^2\dot{r}}$

$$E = \frac{Z_1 Z_2 e^2}{4\pi\varepsilon_o r} + \frac{1}{2}m\dot{r}^2 + \frac{1}{2}\frac{L^2}{mr^2} \rightarrow \dot{r} = \sqrt{\frac{2E}{m} - \frac{Z_1 Z_2 e^2}{2m\pi\varepsilon_o r} - \frac{L^2}{m^2 r^2}}$$

We can write a differential equation as follows:

$$d\theta = \frac{Ldr}{mr^2\dot{r}} = \frac{Ldr}{mr^2\sqrt{\frac{2E}{m} - \frac{Z_1 Z_2 e^2}{2m\pi\varepsilon_o r} - \frac{l^2}{m^2 r^2}}}$$

Then we integrate by substituting $y = \frac{1}{r} \rightarrow dy = -\frac{dr}{r^2}$

$$\int d\theta = \int \frac{-dy}{\sqrt{\frac{2Em}{L^2} - \frac{Z_1 Z_2 e^2 m}{2\pi\varepsilon_o L^2} y - y^2}}$$

Completing a square in the denominator:

$$\frac{2Em}{L^2} - \frac{Z_1 Z_2 e^2 m}{2\pi\varepsilon_o L^2} y - y^2 = \frac{2Em}{L^2} + \left(\frac{Z_1 Z_2 e^2 m}{4\pi\varepsilon_o L^2}\right)^2 - \left(\frac{Z_1 Z_2 e^2 m}{4\pi\varepsilon_o L^2} + y\right)^2$$

We solve the differential equation with the substitution:

$$\left(\frac{Z_1 Z_2 e^2 m}{4\pi\varepsilon_o L^2} + y\right) = \sqrt{\frac{2Em}{L^2} + \left(\frac{Z_1 Z_2 e^2 m}{4\pi\varepsilon_o L^2}\right)^2} \sin\alpha \rightarrow dy = \sqrt{\frac{2Em}{L^2} + \left(\frac{Z_1 Z_2 e^2 m}{4\pi\varepsilon_o L^2}\right)^2} \cos\alpha d\alpha$$

$$\int d\theta = -\int \frac{dy}{\sqrt{\frac{2Em}{L^2} - \frac{Z_1 Z_2 e^2 m}{2\pi\varepsilon_o L^2} y - y^2}} = -\int d\alpha$$

When integrating we could go in θ from π, which corresponds to $r \rightarrow \infty$ to $\theta = \phi$ at closest approach, later we can find the angle of scattering by $\Omega = \pi - 2\phi$, so:

$\phi = \alpha_{max} - \alpha_{min}$

The limits of the integral are:

For $r = r_{minimum}$

$$\frac{\left(\frac{Z_1Z_2e^2m}{4\pi\varepsilon_o L^2}+\frac{2E}{\frac{Z_1Z_2e^2}{4\pi\varepsilon_o}+\sqrt{\left(\frac{Z_1Z_2e^2}{4\pi\varepsilon_o}\right)^2+2E\frac{L^2}{m}}}\right)}{\sqrt{\frac{2Em}{L^2}+\left(\frac{Z_1Z_2e^2m}{4\pi\varepsilon_o L^2}\right)^2}}=\sin\alpha$$

$$\sin\alpha=\frac{2E+\frac{m}{L^2}\left\{\left(\frac{Z_1Z_2e^2}{4\pi\varepsilon_o}\right)^2+\frac{Z_1Z_2e^2}{4\pi\varepsilon_o}\sqrt{\left(\frac{Z_1Z_2e^2}{4\pi\varepsilon_o}\right)^2+2E\frac{L^2}{m}}\right\}}{\frac{Z_1Z_2e^2}{4\pi\varepsilon_o}\sqrt{\frac{2Em}{L^2}+\left(\frac{Z_1Z_2e^2m}{4\pi\varepsilon_o L^2}\right)^2}+\left(\frac{Z_1Z_2e^2}{4\pi\varepsilon_o}\right)^2\frac{m}{L^2}+2E}=1\rightarrow\alpha=\frac{\pi}{2}$$

For $r\rightarrow\infty$

$$\frac{\left(\frac{Z_1Z_2e^2m}{4\pi\varepsilon_o L^2}+\frac{1}{\infty}\right)}{\sqrt{\frac{2Em}{L^2}+\left(\frac{Z_1Z_2e^2m}{4\pi\varepsilon_o L^2}\right)^2}}=\sin\alpha\rightarrow\sin\alpha=\frac{\frac{Z_1Z_2e^2m}{4\pi\varepsilon_o L^2}}{\sqrt{\frac{2Em}{L^2}+\left(\frac{Z_1Z_2e^2m}{4\pi\varepsilon_o L^2}\right)^2}}$$

$$\alpha=\sin^{-1}\frac{\frac{Z_1Z_2e^2m}{4\pi\varepsilon_o L^2}}{\sqrt{\frac{2Em}{L^2}+\left(\frac{Z_1Z_2e^2m}{4\pi\varepsilon_o L^2}\right)^2}}=\tan^{-1}\frac{\frac{Z_1Z_2e^2m}{4\pi\varepsilon_o L^2}}{\sqrt{\frac{2Em}{L^2}}}=\tan^{-1}\frac{Z_1Z_2e^2m}{4\pi\varepsilon_o L\sqrt{2Em}}$$

The scattering angle $\Omega=\pi-2\phi=2\tan^{-1}\frac{Z_1Z_2e^2m}{4\pi\varepsilon_o L\sqrt{2Em}}$

Substituting: $L=mvb=mb\sqrt{2E/m}$

$$\tan\left(\frac{\Omega}{2}\right)=\frac{Z_1Z_2e^2m}{4\pi\varepsilon_o mb\sqrt{2E/m}\sqrt{2Em}}=\frac{Z_1Z_2e^2}{4\pi\varepsilon_o b2E}\rightarrow\qquad b=\frac{Z_1Z_2e^2}{8\pi\varepsilon_o E}\cot\left(\frac{\Omega}{2}\right)$$