Modern Physics

Addition of velocities

Consider a frame of reference K' moving in the frame of reference K with a velocity *v* in the x-direction. Then, the velocities of an object in K transform to K' using the equations

$$u'_x = \frac{u_x - v}{1 - \frac{v u_x}{c^2}} \quad \text{and} \quad u'_y = \frac{u_y}{\gamma\left(1 - \frac{v u_x}{c^2}\right)}$$

Also useful are the equations to transform velocities in K' to K, using these equations

$$u_x = \frac{u'_x + v}{1 + \frac{v u'_x}{c^2}} \quad \text{and} \quad u_y = \frac{u'_y}{\gamma\left(1 + \frac{v u'_x}{c^2}\right)}$$

Problem 1.- An atom moving at speed 0.3c emits an electron along the same direction with speed 0.6c in the internal rest frame of the atom. The speed of the electron in the lab frame is equal to:

(A) 0.25 c (B) 0.51 c (C) 0.66 c (D) 0.76 c (E) 0.90 c

Solution: we use the addition of velocity equation:

$$\mathrm{u_x} = \frac{\mathrm{u'_x + v}}{1 + \frac{\mathrm{vu'_x}}{\mathrm{c^2}}} = \frac{0.3c + 0.6c}{1 + \frac{0.3c \times 0.6c}{c^2}} = \frac{0.9c}{1 + 0.18} = 0.76c$$

Answer: **D**

Problem 1a.- An atom moving at a speed of 0.5c emits an electron along the same direction with speed 0.8c in the internal rest frame of the atom. The speed of the electron in the lab frame is equal to:

Solution: Similar to the previous problem:

$$\mathrm{u_x} = \frac{\mathrm{u'_x + v}}{1 + \frac{\mathrm{vu'_x}}{\mathrm{c^2}}} = \frac{0.5 + 0.8\mathrm{c}}{1 + \frac{(0.8\mathrm{c})(0.5\mathrm{c})}{\mathrm{c^2}}} = 0.93\ \mathrm{c}$$

Problem 2.- A tube of water is traveling at 1/2 c relative to the lab frame when a beam of light traveling in the same direction as the tube enters it. What is the speed of light in the water relative to the lab frame? (The index of refraction of water is 4/3.)

(A) $\frac{1}{2}c$ (B) $\frac{2}{3}c$ (C) $\frac{5}{6}c$ (D) $\frac{10}{11}c$ (E) c

Solution: we use the addition of velocity equation noticing that the speed of light in the frame of reference of the tube is 3/4c, so:

$$u_x = \frac{u'_x + v}{1 + \frac{vu'_x}{c^2}} = \frac{\frac{3}{4}c + \frac{1}{2}c}{1 + \frac{\frac{3}{4}c \times \frac{1}{2}c}{c^2}} = \frac{10}{11}c$$

Answer: **D**

Problem 2a.- A crystal of index of refraction n=1.75 is traveling at 0.8c relative to the lab frame when a beam of light traveling in the same direction as the crystal enters it. What is the speed of light in the crystal relative to the lab frame?

Solution: Similar to the previous problem, the speed of light in the frame of reference of the crystal is $u'_x = \frac{c}{n} = \frac{c}{1.75}$

And the crystal is moving with speed $v = 0.8c$, so to find u_x we use:

$$u_x = \frac{u'_x + v}{1 + \frac{vu'_x}{c^2}} = \frac{\frac{c}{1.75} + 0.8c}{1 + \frac{(0.8c)(c/1.75)}{c^2}} = \mathbf{0.941\ c}$$

Problem 3.- You build a rocket that accelerates to a final velocity of 0.5c. When traveling at this velocity it emits two identical particles one forward and the other backward with speeds of 0.5c with respect to the rocket. What are the speeds of the particles from the point of view of an observer on Earth?

Solution: This is a typical problem of addition of velocities. Consider the rocket to move with the frame of reference K'. Then:

a) When the particle is emitted forward, $u'_x = 0.5c$, in which case:

$$u_x = \frac{u'_x + v}{1 + \frac{u'_x v}{c^2}} = \frac{0.5c + 0.5c}{1 + \frac{(0.5c)(0.5c)}{c^2}} = \frac{c}{1.25} = 0.8c$$

b) When the particle is emitted backward, $u'_x = -0.5c$ in which case:

$$u_x = \frac{u'_x + v}{1 + \frac{u'_x v}{c^2}} = \frac{-0.5c + 0.5c}{1 + \frac{(-0.5c)(0.5c)}{c^2}} = 0$$

Problem 4.- An observer *O* at rest midway between two sources of light at $x = 0$ and $x = 10$ m observes the two sources to flash simultaneously. According to a second observer *O'*, moving at a constant speed parallel to the *x*-axis, one source of light flashes 13 ns before the other. Which of the following gives the speed of *O'* relative to *O*?

(A) 0.13c
(B) 0.15c
(C) 0.36c
(D) 0.53c
(E) 0.62c

Solution: The two flashes in frame of reference K, where observer O is at rest, are observed at the coordinates:

$$x_1 = 0 \quad y_1 = 0 \quad z_1 = 0 \quad t_1 = 0$$
$$x_2 = 10 \quad y_2 = 0 \quad z_2 = 0 \quad t_2 = 0$$

If the other observer is moving with velocity v the coordinates of the two events will be:

$$x_1' = 0 \quad y_1' = 0 \quad z_1' = 0 \quad t_1' = 0$$
$$x_2' = \gamma 10 \quad y_2' = 0 \quad z_2' = 0 \quad t_2' = \gamma\left(-\beta\frac{10}{c}\right)$$

According to the problem:

$$t_2' - t_1' = \gamma\left(-\beta\frac{10}{c}\right) = -13ns$$

Solving for v we find

$$\gamma\left(-\beta\frac{10}{c}\right) = -13ns \rightarrow \gamma\beta = 0.39 \rightarrow \beta = 0.39\sqrt{1-\beta^2} \rightarrow \beta = \frac{0.39}{\sqrt{1+0.39^2}} = 0.36$$

Answer: (C)

Problem 5.- A proton and an antiproton are moving toward each other in a head-on collision at speeds of 0.8c with respect to the lab frame, how fast are they moving with respect to each other?

Solution: To solve this problem we can imagine the collision from the point of view of the proton. Let's call this reference frame K. The antiproton is moving with a velocity U'=-0.8c in a frame of reference K' that itself is moving with a velocity v=-0.8c with respect to K. All we need to do is add the velocities relativistically:

$$U_x = \frac{U_x + v}{1 + \frac{U_x v}{c^2}} = \frac{-0.8c - 0.8c}{1 + 0.64} = \frac{-1.6c}{1.64} \rightarrow U_x = -0.9756c$$

Problem 5a.- Two protons are moving toward each other in a head-on collision. If each has a speed of c/2 with respect to the lab frame, how fast are they moving with respect to each other?

Solution: We use the equation of addition of velocities to get:

$$u_x = \frac{u'_x + v}{1 + \frac{vu'_x}{c^2}} = \frac{-c/2 - c/2}{1 + \frac{(-0.5c)(-0.5c)}{c^2}} = \frac{-c}{1.25} = -0.8c$$

Problem 6.- Three stellar objects A, B and C are aligned and moving away from each other. From the point of view of B, A is receding at a speed 0.6c and C is also receding at 0.6c in the opposite direction. How fast are B and C moving with respect to A?

Solution: The velocity of B from the point of view of A will be the same in magnitude.

$$v_B = 0.6c$$

For the case of C we have to add the velocities relativistically:

$$v_C = \frac{0.6c + 0.6c}{1 + \frac{(0.6c)(0.6c)}{c^2}} = \frac{1.2c}{1.36} \rightarrow \quad v_C = 0.8824c$$

Problem 7.- Rocket A leaves a space station with a speed of 0.826c and later rocket B leaves in the same direction with speed 0.635c. Find the speed of rocket A as seen from rocket B.