Modern Physics

Doppler Effect

Relativistic Doppler Effect: $f_{detected} = f_{source}\sqrt{\dfrac{1-\beta}{1+\beta}}$ for a receding source

Problem 1.- An object that is approaching the Milky Way at 0.6c emits infrared light with a wavelength of 850 nm. Calculate the wavelength of that spectral line as observed from Earth.

Solution: The relation between frequencies and wavelengths is inverse. Then we get

$$\lambda_{detected} = \lambda_{source}\sqrt{\frac{1-0.6}{1+0.6}} = \mathbf{425\ nm}$$

Problem 2.- If a light source is approaching you, will it look redder (lower frequency) or bluer (higher frequency).

Solution: If the light source is approaching it will look bluer. That is why this phenomenon is called blue shift.

Problem 3.- An astronomer observes a star in the constellation Centaurus. She determines that the star has not changed its position in the sky in many years, so she concludes that if the star is moving, the motion has to be radial (along the line of sight that we have from Earth). She then uses a spectrometer to analyze its light spectrum and finds that the Balmer red line, which on Earth is at 656.5 nm, is now at 659.0 nm. Calculate the star velocity.

Solution: The Doppler equation for a source that is receding is:

$$f = \sqrt{\frac{1-\beta}{1+\beta}}f_o \quad \text{so,} \quad \frac{c}{\lambda} = \sqrt{\frac{1-\beta}{1+\beta}}\frac{c}{\lambda_o} \quad \rightarrow \frac{\lambda_o}{\lambda} = \sqrt{\frac{1-\beta}{1+\beta}}$$

We can square both sides and solve for beta as follows:

$$\frac{\lambda_o^{\,2}}{\lambda^2} = \frac{1-\beta}{1+\beta} \quad \rightarrow \quad \frac{\lambda_o^{\,2}}{\lambda^2}(1+\beta) = 1-\beta \quad \rightarrow \quad \frac{\lambda_o^{\,2}}{\lambda^2} + \frac{\lambda_o^{\,2}}{\lambda^2}\beta = 1-\beta$$

$$\frac{\lambda_o^{\,2}}{\lambda^2}\beta + \beta = 1 - \frac{\lambda_o^{\,2}}{\lambda^2} \quad \rightarrow \quad \beta\left(\frac{\lambda_o^{\,2}}{\lambda^2} + 1\right) = 1 - \frac{\lambda_o^{\,2}}{\lambda^2}$$

$$\beta = \frac{1 - \dfrac{\lambda_o^{\,2}}{\lambda^2}}{1 + \dfrac{\lambda_o^{\,2}}{\lambda^2}} = \beta = \frac{1 - \dfrac{656.5^2}{659.0^2}}{1 + \dfrac{656.5^2}{659.0^2}} = 0.003801$$

So, the star is indeed receding (otherwise we would have gotten a negative result for beta) and its velocity is:

$$\mathrm{v} = 0.003801\mathrm{c} = 0.003801(3\times10^{8}\mathrm{m/s}) = 1.14\times10^{6}\ \mathrm{m/s}$$

Problem 4.- The sun, like other stars, emits red light from atomic hydrogen in a transition from the 3rd to the 2nd level (it is called the H_α spectral line) at 6,562.8 Å wavelength.

A commercial H_α filter has a bandwidth of 0.7Å, meaning that it only allows light in the range 6,562.1 Å to 6,563.5 Å to be observed.

Think of this hypothetical case: When looking at a solar flare with a telescope equipped with the filter it suddenly disappears from view. If the flare was coming towards you, estimate the speed of the hydrogen atoms at the time of their disappearance.

Solution: If the source of light is coming towards us, the wavelength will be shorter according to the Doppler shift:

$$\lambda_{\text{detected}} = \lambda_{\text{source}}\sqrt{\frac{1-\beta}{1+\beta}} \to \frac{\lambda^2{}_{\text{detected}}}{\lambda^2{}_{\text{source}}} = \frac{1-\beta}{1+\beta} \to \beta = \frac{\lambda^2{}_{\text{source}} - \lambda^2{}_{\text{detected}}}{\lambda^2{}_{\text{source}} + \lambda^2{}_{\text{detected}}}$$

According to the specs of the filter the cutoff is 6,562.1 Å, this is the wavelength when the flare would disappear and the source wavelength is 6,562.8 Å, so:

$$\beta = \frac{\lambda^2{}_{\text{source}} - \lambda^2{}_{\text{detected}}}{\lambda^2{}_{\text{source}} + \lambda^2{}_{\text{detected}}} = \frac{6562.8^2 - 6{,}562.1^2}{6562.8^2 + 6{,}562.1^2} = 1.07\times10^{-4}$$

Then, the speed of the flare is $v = 1.07\times10^{-4}c = 3.21\times10^{4}\,m/s$

Alternative solution: If β is small you can approximate $\sqrt{\frac{1-\beta}{1+\beta}} \approx 1-\beta$, so in the problem:

$$\lambda_{\text{detected}} = \lambda_{\text{source}}\sqrt{\frac{1-\beta}{1+\beta}} \to \lambda_{\text{detected}} \approx \lambda_{\text{source}}(1-\beta) \to \beta \approx \frac{\lambda_{\text{source}} - \lambda_{\text{detected}}}{\lambda_{\text{source}}} = \frac{0.7}{6562.8} = 1.07\times10^{-4}$$

Problem 5.- A distant galaxy is observed to have its hydrogen-β line shifted to a wavelength of 580 nm, away from the laboratory value of 434 nm. Which of the following gives the approximate velocity of recession of the distant galaxy?

(A) 0.28*c*
(B) 0.53*c*
(C) 0.56*c*
(D) 0.75*c*
(E) 0.86*c*

Solution: According to the Doppler Effect:

$$\lambda_{\text{detected}} = \lambda_{\text{source}}\sqrt{\frac{1+\beta}{1-\beta}} \rightarrow \beta = \frac{1-\left(\lambda_{\text{detected}}/\lambda_{\text{source}}\right)^2}{1+\left(\lambda_{\text{detected}}/\lambda_{\text{source}}\right)^2} = 0.28 \;\; \textbf{answer (A)}$$

Problem 6.- Atomic hydrogen emits photons of $\lambda = 656.6$ nm due to the electronic transition n=3 to 2. Calculate the wavelength that we detect of such emission from an atom that is moving away from us at $\mathrm{v} = \frac{15}{113}\mathrm{c}$

Solution: If the source of light is moving away, the wavelength will be longer according to the Doppler shift:

$$\lambda_{\text{detected}} = \lambda_{\text{source}}\sqrt{\frac{1+\beta}{1-\beta}} = 656.6\sqrt{\frac{1+15/113}{1-15/113}} = \textbf{750.4 nm}$$

Problem 6a.- Atomic hydrogen emits photons of 21cm wavelength due to the hyperfine transition. Calculate the wavelength that we detect of such emission from an atom that is moving away from us at v=0.5c.

Solution: If the source of light is moving away, the wavelength will be longer according to the Doppler shift:

$$\lambda_{\text{detected}} = \lambda_{\text{source}}\sqrt{\frac{1+\beta}{1-\beta}} = 21cm\sqrt{\frac{1+0.5}{1-0.5}} = 21cm\sqrt{3} = \textbf{36.3 cm}$$

Problem 6b.- Atomic hydrogen emits photons of $\lambda = 0.21\mathrm{m}$ due to the hyperfine transition. Calculate the wavelength that we detect of such emission from a star that is moving away from us at $\mathrm{v} = \frac{13}{85}\mathrm{c}$

Solution: If the source of light is moving away, the wavelength will be longer according to the Doppler shift:

$$\lambda_{\text{detected}} = \lambda_{\text{source}}\sqrt{\frac{1+\beta}{1-\beta}} = 0.21\sqrt{\frac{1+13/85}{1-13/85}} = \textbf{0.245 m}$$

Problem 7.- A star is observed to emit the red line from atomic hydrogen (called the H_α spectral line) at 661.6 nm wavelength instead of the normal 656.3 nm wavelength. Assuming only radial velocity, is the star approaching us or receding from us? How fast is it moving with respect to us?

Solution: Since the wavelength is longer the light is red-shifted, which means that the star is moving away from us. To calculate the velocity we use the Doppler equation:

$$f_{\text{detected}} = f_{\text{source}}\sqrt{\frac{1-\beta}{1+\beta}} \rightarrow \frac{1}{\lambda_{\text{detected}}} = \frac{1}{\lambda_{\text{source}}}\sqrt{\frac{1-\beta}{1+\beta}}$$

$$\rightarrow \beta = \frac{\lambda^2{}_{\text{detected}} - \lambda^2{}_{\text{source}}}{\lambda^2{}_{\text{source}} + \lambda^2{}_{\text{detected}}} = \frac{661.6^2 - 656.3^2}{661.6^2 + 656.3^2} = 0.00804$$

So the speed of the star is: $v = \beta c = 0.00804(3\times10^8 m/s) =$ **2.4×10^6 m/s**

Problem 8.- If a radio source is leaving the solar system at a speed of 0.92c and emitting at a frequency of 400kHz, what will be the frequency detected on earth?

Solution: Beta is 0.92 so:

$$f = \frac{\sqrt{1-\beta}}{\sqrt{1+\beta}} f_o = \frac{\sqrt{1-0.92}}{\sqrt{1+0.92}} 400kHz \rightarrow f = \mathbf{82\ kHz}$$

Problem 9.- A detector observes a star emitting infrared radiation with the wavelengths 1334.5nm, 1138.9nm and 1046.1nm that are suspected to be from the Paschen series. Identify the corresponding lines and calculate the speed of the star.

Solution: The known wavelengths of the Paschen series can be calculated using the Rydberg equation. These are the results:

n	k	lambda (nm)
3	4	1875.6
3	5	1282.2
3	6	1094.1
3	7	1005.2
3	8	954.9

The measured wavelengths are different. That is fine, we know that if the emitter is moving, the observed wavelengths will be different due to the Doppler Effect.

To find out how fast the emitter is moving we can use the Doppler equation.

Note: This problem has a subtlety, which is that the first wavelength measured does not correspond to the first line in the Paschen series, possibly because the detector cannot work at the redshifted wavelength.

If we take the ratio between the measured and calculated wavelengths we get a constant, as follows:

n	k	lambda (nm) Calculated	lambda (nm) Measured	Ratio
3	4	1875.6		
3	5	1282.2	1334.5	1.0408
3	6	1094.1	1138.9	1.0409
3	7	1005.2	1046.1	1.0407
3	8	954.9		

The measured wavelengths are longer than the ones calculated, so the star is receding (going away from us). The Doppler shift gives:

$$1.0408 = \sqrt{\frac{1+\beta}{1-\beta}} \rightarrow \beta = \frac{1.0408^2 - 1}{1.0408^2 + 1} = 0.04 \rightarrow \text{v} = \mathbf{1.2\times10^7\ m/s}$$