Modern Physics

Energy and Momentum

Relativistic Momentum: $p = \gamma mv$

Kinetic energy: $K.E. = (\gamma - 1)mc^2$

Auxiliary equation: $E^2 = p^2c^2 + m^2c^4$ where m is the rest mass and E is total energy.

Problem 1.- What has more mass at rest:

i) A sodium ^{23}Na nucleus

ii) 11 protons and 12 neutrons

Solution: The nucleus has less mass. The difference in mass times c^2 is the binding energy.

Problem 2.- Calculate the total energy of a muon (whose mass at rest is $106\text{MeV}/c^2$) that is moving at a speed of 0.97c.

Solution: $\text{Energy} = \gamma mc^2$

$$\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{1}{\sqrt{1-\frac{(0.97c)^2}{c^2}}} = \frac{1}{\sqrt{1-0.97^2}} = \frac{1}{\sqrt{1-0.9409}} = \frac{1}{\sqrt{0.0591}} = 4.11345$$

So,

$$\text{Energy} = \gamma mc^2 = 4.11345(106\text{MeV}/c^2)c^2 = 436MeV$$

Problem 3.- If the kinetic energy of a particle of mass m is equal to its rest energy mc^2, how much is its momentum?

(A) $p = mc$ (B) $p = \sqrt{2}mc$ (C) $p = \sqrt{3}mc$

(D) $p = 2mc$ (E) $p = 0.5mc$

Solution: we use the ancillary equation:

$$E^2 = p^2c^2 + m^2c^4 \rightarrow (mc^2 + mc^2)^2 = p^2c^2 + m^2c^4 \rightarrow 4m^2c^4 = p^2c^2 + m^2c^4$$

$$\rightarrow 3m^2c^4 = p^2c^2 \rightarrow p = \sqrt{3}mc$$

answer: **C**

Problem 3a.- What is the speed of a proton whose kinetic energy is three times its rest energy?

Solution: According to the problem $K.E.=3mc^2$, so:

$$K.E. = (\gamma - 1)mc^2 \rightarrow 3mc^2 = (\gamma - 1)mc^2 \rightarrow \gamma = 4$$

But, $\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$, so: $4 = \frac{1}{\sqrt{1 - v^2/c^2}} \rightarrow 1 - v^2/c^2 = \frac{1}{16} \rightarrow v^2/c^2 = \frac{15}{16}$

$$v/c = \frac{\sqrt{15}}{4} \rightarrow v = \frac{\sqrt{15}}{4} c$$

Problem 3b.- If the kinetic energy of a particle of mass m is equal to four times its rest energy, then the magnitude of the particle's relativistic momentum is:

Solution: According to the problem K.E. = $4mc^2$, so the total energy is E = $5mc^2$ and using the ancillary equation:

$$E^2 = p^2c^2 + m^2c^4 \rightarrow (5mc^2)^2 = p^2c^2 + m^2c^4 \rightarrow 25m^2c^4 = p^2c^2 + m^2c^4$$

Solving for p:

$$p^2c^2 = 24m^2c^4 \rightarrow p = \sqrt{24}mc$$

Problem 3c.- If the kinetic energy of a particle of mass m is equal to twice its rest energy, then the magnitude of the particle's relativistic momentum is:

Solution: According to the problem K.E. = $2mc^2$, so the total energy is E = $3mc^2$ and using the ancillary equation:

$$E^2 = p^2c^2 + m^2c^4 \rightarrow (3mc^2)^2 = p^2c^2 + m^2c^4 \rightarrow 9m^2c^4 = p^2c^2 + m^2c^4$$

Solving for p:

$$p^2c^2 = 8m^2c^4 \rightarrow p = \sqrt{8}mc$$

Problem 4.- How much energy do you need to give an electron to accelerate it from rest to 0.99c? Give your answer in electron volts.

(A) $3.1MeV$ (B) $4.1MeV$ (C) $5.1MeV$
(D) $6.1MeV$ (E) $7.1MeV$

Solution: we calculate the kinetic energy:

$$\text{K.E.} = (\gamma - 1)\text{mc}^2 = \left(\frac{1}{\sqrt{1\text{-}0.99^2}} - 1\right)\text{mc}^2 = (7.088 - 1) \times 0.511\text{MeV} = 3.1MeV$$

answer: **A**

Problem 5.- A rest mass is 4 grams and is traveling at 3/5 the speed of light when it collides head-on with an identical mass going the opposite direction at the same speed. If the two masses stick together and no energy is radiated away, what is the composite object?

(A) 4g (B) 6.4g (C) 8g (D) 10g (E) 13.3g

Solution: we calculate the total energy before and after:

$$\mathrm{T}E_{before} = \mathrm{T}E_{after} \rightarrow 2\gamma mc^2 = Mc^2 \rightarrow M = 2\gamma m = \frac{2m}{\sqrt{1-(3/5)^2}} = \frac{2\times 4g}{\sqrt{1-0.6^2}} = 10g$$

answer: **D**

Problem 6.- A particle leaving a cyclotron has a total relativistic energy of 10 GeV and a relativistic momentum of 8 GeV/c. What is the rest mass of this particle?

(A) $0.25\frac{GeV}{c^2}$ (B) $1.2\frac{GeV}{c^2}$ (C) $2\frac{GeV}{c^2}$ (D) $6\frac{GeV}{c^2}$ (E) $16\frac{GeV}{c^2}$

Solution: we use the ancillary equation:

$$\mathrm{E}^2 = \mathrm{p}^2\mathrm{c}^2 + \mathrm{m}^2\mathrm{c}^4 \rightarrow (10GeV)^2 = (8\mathrm{GeV/c})^2\mathrm{c}^2 + \mathrm{m}^2\mathrm{c}^4 \rightarrow 100GeV^2 = 64GeV^2 + \mathrm{m}^2\mathrm{c}^4$$

$$\rightarrow 36GeV^2 = \mathrm{m}^2\mathrm{c}^4 \rightarrow m = \frac{6GeV}{c}$$

answer: **D**

Problem 6a.- A particle leaving a cyclotron has a total relativistic energy of 1.3 MeV and a relativistic momentum of 1.2 MeV/c. What is the rest mass of this particle?

Solution: Using the ancillary equation:

$$E^2 = p^2c^2 + m^2c^4 \rightarrow (1.3MeV)^2 = (1.2MeV/c)^2c^2 + m^2c^4$$

Solving for m:

$$1.69MeV^2 = 1.44MeV^2 + m^2c^4 \rightarrow m = \frac{0.5MeV}{c^2}$$

Problem 7.- Demonstrate that if the force is parallel to the velocity of a particle you can write Newton's second law of motion as

$$\vec{F} = \gamma^3 m\vec{a}$$

Solution: According to Newton's second law: $\vec{F} = \frac{d\vec{p}}{dt}$ which is still valid in special relativity, but we need to use the relativistic momentum, not p=mv as in classical physics. If the force and velocity are parallel we can drop the vector notation and treat the problem as a one dimensional problem:

$$F = \frac{dp}{dt} = \frac{d\gamma mv}{dt} = m\frac{d}{dt}\left(\frac{v}{\sqrt{1-v^2/c^2}}\right) = m\frac{d}{dv}\left(\frac{v}{\sqrt{1-v^2/c^2}}\right)\frac{dv}{dt}$$

Notice that we used the chain rule of derivatives and *dv/dt* is the acceleration "*a*", so the force becomes:

$$F = m\frac{d}{dv}\left(\frac{v}{\sqrt{1-v^2/c^2}}\right)a = ma\frac{\sqrt{1-v^2/c^2} - v\frac{-v/c^2}{\sqrt{1-v^2/c^2}}}{1-v^2/c^2} = ma\frac{1-v^2/c^2+v^2/c^2}{(1-v^2/c^2)^{3/2}} =$$

$$= \frac{ma}{(1-v^2/c^2)^{3/2}} = \gamma^3 ma$$

Problem 8.- A particle of mass M decays from rest into two particles. One particle has mass m and the other particle is massless. The momentum of the massless particle is:

$(A)\frac{(M^2-m^2)c}{4M}$

$(B)\frac{(M^2-m^2)c}{2M}$

$(C)\frac{(M^2-m^2)c}{M}$

$(D)\frac{2(M^2-m^2)c}{M}$

$(E)\frac{4(M^2-m^2)c}{M}$

Solution: Conservation of energy:

$$Mc^2 = \sqrt{m^2c^4+p^2c^2} + pc$$

$$Mc^2 - pc = \sqrt{m^2c^4+p^2c^2} \rightarrow M^2c^4 - 2pcMc^2 = m^2c^4$$

$$p = \left(\frac{M^2-m^2}{2M}\right)c$$ **answer (B)**

Problem 9.- What is the kinetic energy of an electron whose speed is $v = \frac{79}{137}c$?

Solution: $K.E. = (\gamma - 1)mc^2 = \left(\frac{1}{\sqrt{1-\left(\frac{79}{137}\right)^2}} - 1 \right) mc^2 = 0.224mc^2$

In joules: $K.E. = 1.83 \times 10^{-14} J$, and in eV: $K.E. = 115 keV$

Problem 9a.- Calculate the kinetic energy of an electron moving at a speed c/15. Should we use relativistic or classical equations? What accelerating voltage is necessary to reach that speed?

Solution: The kinetic energy can be calculated classically. Since the ratio v/c is 1/15, the correction is very small. However, let us calculate both ways:

$$\text{K.E.}_{\text{Classical}} = \frac{1}{2}\text{mv}^2 = \frac{1}{2}(9.11\times10^{-31}\text{kg})(2.00\times10^7\text{m/s})^2 = 1.822\times10^{-16}\text{J}$$

$$\text{K.E.}_{\text{Relativistic}} = (\gamma - 1)\text{mc}^2 = \left(\frac{1}{\sqrt{1-\frac{\text{v}^2}{\text{c}^2}}} - 1 \right)\text{mc}^2$$

$$\text{K.E.}_{\text{Relativistic}} = \left(\frac{1}{\sqrt{1-\frac{1}{15^2}}} - 1 \right)(9.11\times10^{-31})(3.00\times10^8\text{m/s})^2 = 1.828\times10^{-16}\text{J}$$

To calculate the voltage to get this energy we divide by the charge, since

$$(\text{Voltage})\times(\text{Charge}) = \text{K.E.}$$

$$\text{Voltage} = \frac{1.82\times10^{-16}\text{J}}{1.60\times10^{-19}\text{C}} = \mathbf{1{,}140\ V}$$

Problem 10.- Calculate the mass of a particle whose momentum is 10^{-16} kgm/s when its speed is 0.92c.

Solution: Recall that for speeds close to the speed of light we should use relativistic momentum:

$$p = \gamma mv$$

So: $$m = \frac{p}{\gamma v} = \frac{p\sqrt{1-\frac{v^2}{c^2}}}{v} = \frac{10^{-16} kgm/s\sqrt{1-0.92^2}}{0.92(3\times10^8 m/s)} \rightarrow m = 1.42\times10^{-25} kg$$

Problem 11.- Calculate the speeds of an electron and a positron that have kinetic energies of 9GeV and 3.1GeV respectively.

Solution: We are given the kinetic energy and that allows us to find gamma:

$$K.E. = mc^2(\gamma - 1) \rightarrow \gamma = \frac{K.E.}{mc^2} + 1$$

So, for the electron:

$$\gamma = \frac{9GeV}{0.511MeV} + 1 = 17613.5$$

And for the positron (whose mass is the same as the electron)

$$\gamma = \frac{3.1GeV}{0.511MeV} + 1 = 6067.5$$

Knowing gamma is enough to find the speed since:

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \rightarrow v = c\sqrt{1 - \frac{1}{\gamma^2}}$$

$$v_{electron} = 0.999999998c$$
$$v_{positron} = 0.999999986c$$

Problem 12.- Complete the statement:

The primary source of the Sun's energy is a series of thermonuclear reactions in which the energy produced is c^2 times the mass difference between ___________________ .

Solution: a helium atom and four hydrogen atoms.

Problem 13.- The ^{238}U nucleus (isotope of uranium) has a binding energy of 7.6 MeV per nucleon. If it were to fission into two equal fragments, each would have a kinetic energy of just over 100 MeV. From this, estimate the binding energy per nucleon of the fragments.