Modern Physics

Lorentz transformation

These equations transform coordinates from frame K to K', where frame K' moves in the positive x-direction with speed v and at t=t'=0 the two frames coincide.

$$x' = \gamma(x - \beta ct) \qquad y' = y \qquad z' = z \qquad t' = \gamma(t - \beta\frac{x}{c})$$

Where $\beta = \frac{v}{c}$ and $\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$

Problem 1.- Consider the events:

A= (0, 0, 0, 0)

B= (100m, 0, 0, 1μs)

a) Find a frame of reference in which the two events happen in the same place if it is possible.

b) Find a frame of reference in which the two events happen at the same time if it is possible.

Solution:

a) If the two events happen in the same place: $x'_A = x'_B$ which means:

$$\gamma(x_A - \beta ct_A) = \gamma(x_B - \beta ct_B) \rightarrow x_B - \beta ct_B = 0 \rightarrow \beta = \frac{x_B}{ct_B} = \frac{100\text{m}}{3\times10^8\,\text{m/s}(1\times10^{-6}\text{s})} = \frac{1}{3}$$

This means that v=c/3

b) If the two events happen at the same time: $t'_A = t'_B$ which means:

$$\gamma(t_A - \beta\frac{x_A}{c}) = \gamma(t_B - \beta\frac{x_B}{c}) \rightarrow t_B - \beta\frac{x_B}{c} = 0 \rightarrow \beta = \frac{ct_B}{x_B} = \frac{3\times10^8\,\text{m/s}(1\times10^{-6}\text{s})}{100\text{m}} = 3$$

This means that v=3c which is not possible.

Problem 2.- The frame of reference K' moves with respect to another frame K in the x direction (the x and x' axis are parallel) with speed 0.99c. At t=0, t'=0 and the origins of both frames coincide. Calculate the coordinates in frame K' of an event that happens in frame K at t=2ns, x=1m, y=2m and z=3m

Solution:
Using the Lorentz transformations, we get:

$$\beta = \frac{v}{c} = 0.99 \qquad \text{and} \qquad \gamma = \frac{1}{\sqrt{1 - v^2/c^2}} = \frac{1}{\sqrt{1 - 0.99^2}} = 7.09$$

$$x' = 7.09(1\text{m} - 0.99(0.6\text{m})) = 2.9m$$
$$y' = 2\text{m}$$
$$z' = 3\text{m}$$
$$t' = \gamma(2\text{ns} - 0.99\frac{1\text{m}}{\text{c}}) = -9.2ns$$

Problem 2a.- Frame of reference K' moves with respect to K in the x direction (the x and x' axis are parallel) with speed 0.96c. At t=0, t'=0 and the origins of both frames coincide. Calculate the coordinates in frame K' of an event that happens in frame K at t=2.8ns, x=1.4m, y=3m and z=4m

Solution: Calculating β and γ we get:

$\beta = \frac{\text{v}}{\text{c}} = 0.96$ and $\gamma = \frac{1}{\sqrt{1-0.96^2}} = 3.57$

In the Lorentz transformation:

$$x' = \gamma(x - \beta ct) = 3.57(1.4 - 0.96 \times 3 \times 10^8 \times 2.8 \times 10^{-9}) = 2.12m$$
$$y' = y = 3\text{m}$$
$$z' = z = 4\text{m}$$
$$t' = \gamma(t - \beta\frac{x}{c}) = 3.57(2.8 \times 10^{-9} - 0.96 \times \frac{1.4}{3 \times 10^8}) = -6ns$$

Problem 3.- In inertial frame K, two events occur at the same time and 24c-minutes apart in space. In inertial frame K', the same events occur at 25c-minutes apart. What is the time interval between the events in K'?

Solution: Knowing that the space-time interval is an invariant we can write:

$$\Delta s^2 = \Delta x^2 - c^2\Delta t^2 = \Delta x'^2 - c^2\Delta t'^2$$

But $\Delta t = 0$ and $\Delta x = 24c\,\text{min}$. We also know that $\Delta x' = 25c\,\text{min}$, so:

$$(24c\,\text{min})^2 - c^2(0)^2 = (25c\,\text{min})^2 - c^2\Delta t'^2 \rightarrow \Delta t' = 7\,\text{min}$$

Problem 4.- In inertial frame K events 1 and 2 occur at the same time, but 4 kilometers apart on the x-axis. In inertial frame K', which is moving in the x-direction, the events occur 5 kilometers apart. What is the time difference between the events in K'?

Solution: For the frame of reference where the events are simultaneous, we can write

$$X_1 = 0$$
$$X_2 = 4km$$
$$t_1 = 0$$
$$t_2 = 0$$

For the other frame of reference, we have:

$$X_1' = 0$$
$$X_2' = 5km$$
$$t_1' = 0$$
$$t_2' = ?$$

The Lorentz transformation for X_2 is as follows:

$$X_2' = \gamma(X_2 - vt_2)$$

This allows us to find γ:

$$5km = \gamma(4km - 0) \rightarrow \gamma = 5/4$$

Knowing gamma allows us to find v:

$$\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \rightarrow v = c\sqrt{1-\frac{1}{\gamma^2}} \rightarrow v = 0.6c$$

Now that we know gamma and *v,* we can calculate t_2'

$$t_2' = \frac{5}{4}(0 - \frac{(0.6c)(4km)}{c^2}) \rightarrow t_2' = 1\times10^{-5}\text{s}$$

Problem 5.- In a frame of reference K two events occur as follows:

Event1	Event2
$x_1 = a$	$x_2 = 2a$
$y_1 = 0$	$y_2 = 0$
$z_1 = 0$	$z_2 = 0$
$t_1 = 2a/c$	$t_2 = 3a/2c$

Find a frame of reference K' where the events happen simultaneously.

Solution: For the events to be simultaneous, we can consider a frame with velocity in the x-direction and write an equation where the times are equal:

$$t_2' = \gamma(t_2 - \beta\frac{x_2}{c}) = t_1' = \gamma(t_1 - \beta\frac{x_1}{c})$$

$$t_2 - \beta\frac{x_2}{c} = t_1 - \beta\frac{x_1}{c}$$

$$\beta = \frac{t_2 - t_1}{\frac{x_2}{c} - \frac{x_1}{c}} = -0.5$$

We find that K' should be moving in the negative x-direction with half the speed of light.

Problem 6.- Consider the following scenario: In an inertial frame K, events occur A and B separated in time by Δt and in space by Δx. In another inertial frame K', with velocity v in the x-direction relative to K, the two events occur at the same time.
Is that possible? If yes, under what conditions?

(A) Not possible
(B) Possible for any values of Δx and Δt
(C) Possible if $|\Delta x / \Delta t| < c$
(D) Possible if $|\Delta x / \Delta t| > c$
(E) Possible if $|\Delta x / \Delta t| = c$