Modern Physics

Time dilation

$$T = \gamma T_o$$

Problem 1.- Suppose astronauts want to visit a planet that is 20 light years away from earth. If the round trip should take 40 years for the astronauts, what is the minimum speed of the rocket? And how much time would have passed on earth when they come back?
Ignore the acceleration time.

Solution: The astronauts will measure a time of 40 years back and forth, or 20 years each way. This time is what we call "proper time" because the ship is at rest with respect to the astronauts.

$T_o = 40\,years$ Time back and forth, as seen from the ship

The distance from Earth to the other planet is 20 light years (20 years times the speed of light) from the point of view of the Earth. In this reference frame both planets are at rest (neglecting the slow speed of the orbits) so 20 light years is the "proper" distance to the other planet.

$Lo' = (40\,years)c$ Distance back and forth as seen from Earth

From the point of view of the ship, the Earth and the other planet will be moving, so the distance between them will appear contracted by the factor gamma.

$$L = \frac{L_o'}{\gamma}$$

Then, the velocity of the ship from the point of view of the astronauts will be:

$$v = \frac{L}{T_o} = \frac{L_o'}{\gamma T_o}$$

Using the information that we have, we can write this equation as:

$$v = \frac{(40\,years)c}{\gamma 40\,years} = \frac{c}{\gamma} \quad \text{or} \quad \frac{v}{c} = \frac{1}{\gamma} \quad \text{which means that } \frac{v}{c} = \sqrt{1 - \frac{v^2}{c^2}}$$

We can square both sides and solve for the velocity:

$$v = \frac{\sqrt{2}}{2}c = 0.707c = 2.12 \times 10^8 \frac{m}{s}$$

For the second question, recall that the time as measured from earth will be dilated by the factor gamma (which we already know is equal to $\sqrt{2}$):

$$T = \gamma T_o = \sqrt{2}T_o = \sqrt{2} \times 40\,years = 56.6\,years$$

Problem 1a.- A friend of yours travels to Sirius, the second brightest star that we can see, which is 8.6 light years away from the Earth. According to her it took 8.33 years for the whole trip back and forth.

a) Calculate her velocity assuming it was constant on each leg of the trip (ignore acceleration times).
b) How much older are you when she comes back?

Solution: The proper time of the astronaut is 8.33 years and that multiplied by γ is the time from our point of view:

$$T = \gamma T_o \to \frac{2\times 8.6cy}{v} = \gamma 8.33y \to \beta = \frac{2\times 8.6}{8.33\gamma} = \frac{2\times 8.6}{8.33}\sqrt{1-\beta^2}$$

Squaring both sides and solving for β:

$$\beta^2 = \left(\frac{2\times 8.6}{8.33}\right)^2 \left(1-\beta^2\right) \to \beta^2 = \frac{\left(\frac{2\times 8.6}{8.33}\right)^2}{1+\left(\frac{2\times 8.6}{8.33}\right)^2} \to \beta = \sqrt{\frac{\left(\frac{2\times 8.6}{8.33}\right)^2}{1+\left(\frac{2\times 8.6}{8.33}\right)^2}} = 0.9$$

So the velocity of the astronaut is 0.9c.

To find the time that passed on earth we multiply the 8.33y (the proper time) times γ:

$$T = \gamma 8.33y = \frac{8.33y}{\sqrt{1-\beta^2}} = \frac{8.33y}{\sqrt{1-0.9^2}} = 19.11 \text{ years}$$

Problem 1b.- An astronaut is going to Alpha Centauri (4.365 light years away). What speed will be necessary if she wishes to age only 3 years during the round trip? Ignore the acceleration times.

Solution: If her velocity is v, the time of flight from the point of view of the earth will be:

$T = \dfrac{2(4.365cy)}{v}$ where "y" is 1 year and we multiply by 2 to account for the two legs of the trip.

But the proper time is 3 years, so $T_o = 3y$, now according to time dilation $T = \gamma T_o$, so:

$$\frac{2(4.365cy)}{v} = \gamma 3y$$

Solving for v:

$$\frac{2(4.365c)}{v} = \gamma 3 \to \frac{2(4.365c)}{v} = \frac{3}{\sqrt{1-v^2/c^2}} \to \frac{v}{c} = \frac{2\times 4.365\sqrt{1-v^2/c^2}}{3}$$

$$\frac{v^2}{c^2} = 8.468\left(1-\frac{v^2}{c^2}\right) \rightarrow 9.468\frac{v^2}{c^2} = 8.468 \rightarrow \frac{v}{c} = \sqrt{\frac{8.468}{9.468}} = 0.946$$

The necessary speed is then: $v = 2.84\times10^8 m/s$

Problem 2.- The half life of a π^+ meson is $2.6\times10^{-8}s$ in a frame of reference where it is at rest. How long is the half life in the lab frame if it is moving at v=0.8c?

(A) $1.5\times10^{-8}s$ (B) $2.0\times10^{-8}s$ (C) $2.5\times10^{-8}s$
(D) $3.1\times10^{-8}s$ (E) $4.3\times10^{-8}s$

Solution: we use the time dilation equation:

$$\gamma = \frac{1}{\sqrt{1-\mathrm{v}^2/\mathrm{c}^2}} = \frac{1}{\sqrt{1-0.8^2}} = \frac{1}{0.6} = 1.667$$

$$\mathrm{t} = \gamma t_\circ = 1.667\times2.\times10^{-8} = 4.3\times10^{-8}s$$ answer: **E**

Problem 3.- The half life of a π^+ meson is $2.6\times10^{-8}s$. A beam of π^+ mesons is generated at a point 15.6 meters from a detector. Only ½ of the π^+ mesons live to reach the detector. The speed of the π^+ mesons is

(A) $\frac{1}{2}c$ (B) $\sqrt{\frac{2}{5}}c$ (C) $\frac{2}{\sqrt{5}}c$ (D) c (E) $2c$

Solution: we use the time dilation equation, so the time is: $\mathrm{t} = \gamma t_\circ = \frac{2.6\times10^{-8}s}{\sqrt{1-\beta^2}}$

And the distance is 15.6 meters, so the speed is: $\mathrm{v} = \frac{x}{t} = \frac{15.6\sqrt{1-\beta^2}}{2.6\times10^{-8}}$

To simplify the problem we divide by the speed of light:

$$\frac{\mathrm{v}}{c} = \frac{15.6\sqrt{1-\beta^2}}{2.6\times10^{-8}\times3\times10^8} \rightarrow \beta = 2\sqrt{1-\beta^2} \rightarrow \beta^2 = 4-4\beta^2 \rightarrow \beta = \frac{2}{\sqrt{5}}$$ Answer: **C**

Problem 4.- When neutrons are free (not being part of a nucleus) they are unstable with a half-life of 608s. A source of fast neutrons with v = 0.98c emits a pulse of 1,000 neutrons. Find how many will be left in average after traveling 1.79×10^{11} m.

Solution: The time of flight from the point of view of the source is:

$$t = \frac{1.79\times10^{11} m}{0.98(3\times10^{8} m/s)} = 609s$$

But the time from the point of view of the neutrons is only:

$$t_o = \frac{609s}{\gamma} = 609s\sqrt{1-0.98^2} = 121.2\text{s}$$

So, the number of neutrons left is in average:

$$N = 1000\times 2^{-121.2/608} = 871$$

Problem 5.- A beam of 1,000 atoms of unstable americium isotope ^{232}Am decay in 300 seconds so only 125 atoms remain in the beam. Calculate the speed of the beam if their half-life is 79s.

Solution: Since only 1/8 of the initial particles are left, three half-lives passed, that is 237s, but this is the proper time. In the lab frame, the time is dilated to 300s, so:

$$\gamma = \frac{300}{237} \rightarrow \sqrt{1-\beta^2} = \frac{237}{300} \rightarrow \beta = \sqrt{1-\left(\frac{237}{300}\right)^2} = 0.61$$

So the speed is 0.61c

Problem 6.- ^{232}Am is an unstable isotope of americium (an element commonly found in smoke detectors) with a half life of 79s. A pulse of 1,000 fast atoms of ^{232}Am is emitted with v = c/2. Estimate how long it will take for 750 isotopes to decay.

Solution: Since only ¼ of the initial particles is left, two half-lives passed, that is 158s, but this is the proper time. In the lab frame the time is dilated to:

$$t = \gamma t_o = \frac{t_o}{\sqrt{1-\beta^2}} = \frac{158s}{\sqrt{1-0.5^2}} = 182 \text{ s}$$

Problem 7.- In the future (or maybe now, if you are reading this in 100 years?) a spacecraft moves at such a speed that its clocks run at 3/5 the speed on earth. What is the speed of the spacecraft?

Solution: This is a direct application of time dilation. We know that gamma has to be equal to 5/3 so the clock on the ship will run at a speed 3/5 of that of a clock at rest.

$$\gamma = 5/3 = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$$

Solving for v we get: $\frac{3}{5} = \sqrt{1-\frac{v^2}{c^2}} \rightarrow \frac{9}{25} = 1-\frac{v^2}{c^2} \rightarrow \frac{v^2}{c^2} = \frac{16}{25}$ So, $v = 0.8c$