Modern Physics

Bohr model

Energy: $E = -\frac{Z^2}{n^2}\frac{me^4}{32\pi^2\varepsilon_o^2\hbar^2} = -\frac{Z^2}{n^2}E_o$ Where $E_o = 13.6eV$

Radius: $r = \frac{n^2}{Z}a_0$ Where $a_0 = 0.529$ Å is the Bohr radius.

Emission of light: $\frac{hc}{\lambda} = Z^2 E_o\left(\frac{1}{n_f^2} - \frac{1}{n_i^2}\right)$ Where $E_o = 13.6eV$

Rydberg equation: $\frac{1}{\lambda} = \frac{1}{91.14nm}\left(\frac{1}{n_f^2} - \frac{1}{k_i^2}\right)$

Velocity of an electron: $v = \frac{Z}{n}\sqrt{\frac{2E_o}{m}}$ Where $E_o = 13.6eV$

Problem 1.- Consider a helium ion, He^+. Calculate the kinetic energy, radius of the orbit and velocity of the electron in the state with quantum number n=15.

Solution:

Kinetic energy: $KE = \frac{Z^2}{n^2}\times 13.6eV = \frac{2^2}{15^2}\times 13.6eV = 0.24eV$

Radius of the orbit: $r = \frac{n^2}{Z}\times 0.529\dot{\text{A}} = \frac{15^2}{2}\times 0.529\dot{\text{A}} = 59.5\dot{\text{A}}$

Velocity: $v = \frac{Z}{n}\frac{c}{137} = \frac{2}{15}\times\frac{c}{137} = \frac{c}{1027.5} = 292{,}000 m/s$

Problem 2.- What is the minimum photon energy needed to remove the last electron of Li^{++}?

Solution: Li^{++} is a hydrogenic atom because it only posses one electron. We can apply the Bohr model to calculate its ionization energy:

$IP = Z^2 E_o = 3^2(13.6eV) =$ **122.4 eV**

Problem 2a.- What is the minimum energy necessary to remove the last electron from the sextuple ionized nitrogen atom N^{+6}?

Solution: The ion N^{+6} will behave like a hydrogenic atom, so to remove the last electron we will need:

$E = Z^2\frac{13.6eV}{1^2} = 7^2\frac{13.6eV}{1^2} =$ **666.4 eV**

Problem 2c.- If you could remove 91 electrons of the Uranium atom, how much energy would it cost you to remove the last electron?

Solution: We can use the hydrogenic equation in this case (the ion would behave like a hydrogen atom), and the energy would be:

$b = Z^2 E_o = 92^2(13.6eV) =$ **115 keV**

Notice that relativistic effects are important in this case, so this is an approximation.

Problem 3.- In the hydrogen spectrum, the ratio of the wavelengths for Lyman-α radiation (n = 2 to n = 1) to Balmer-α radiation (n = 3 to n = 2) is

Solution: We can use Rydberg's equation to find the wavelengths:

$$\frac{1}{\lambda_1} = \frac{1}{91.14nm}\left(\frac{1}{1^2} - \frac{1}{2^2}\right) = \frac{1}{91.14nm}\left(\frac{3}{4}\right)$$

$$\frac{1}{\lambda_2} = \frac{1}{91.14nm}\left(\frac{1}{2^2} - \frac{1}{3^2}\right) = \frac{1}{91.14nm}\left(\frac{5}{36}\right)$$

Dividing the second equation by the first:

$$\frac{\lambda_1}{\lambda_2} = \frac{\frac{1}{91.14nm}\left(\frac{5}{36}\right)}{\frac{1}{91.14nm}\left(\frac{3}{4}\right)} = \frac{5}{27}$$

Problem 4.- Calculate the energy of the photon emitted if a positronium atom makes a transition from a state with n = 2 to the state with n = 1

Solution: This problem is similar to a Lyman transition, but the energy is only half of that of a hydrogen atom:

$$\Delta E = -\frac{13.6eV}{2}\left(\frac{1}{2^2} - \frac{1}{1^2}\right) = \textbf{5.1 eV}$$

Problem 5.- Calculate the wavelength of the first line in the Balmer series that is invisible to us (because $\lambda < 400nm$).

Solution: We want the wavelength corresponding t the transition k=7 to n=2, the other 4 lines are visible (3 to 2 is red, 4 to 2 is blue-green, 5 to 2 and 6 to 2 are violet, but visible)

$$\frac{1}{\lambda} = \frac{1}{91.14nm}\left(\frac{1}{n_f^2} - \frac{1}{n_i^2}\right) = \frac{1}{91.14nm}\left(\frac{1}{7^2} - \frac{1}{2^2}\right) = \frac{1}{91.14nm}\left(\frac{7^2 - 2^2}{2^2 \times 7^2}\right) = \frac{1}{91.14nm}\left(\frac{45}{196}\right)$$

$$\lambda = 91.14nm\left(\frac{196}{45}\right) = 397nm$$

Problem 5.- Calculate the wavelength of the Paschen line from n=5 to n=3.

Solution: We know the wave is going to be infrared, the wavelength is given by:

$$\frac{1}{\lambda}=Z^2\left(\frac{1}{{n_2}^2}-\frac{1}{{n_1}^2}\right)R=1^2\left(\frac{1}{3^2}-\frac{1}{5^2}\right)R=\frac{16}{225}R\rightarrow\lambda=\frac{225}{16R},$$

Knowing that the value of R is $R=\frac{me^4}{32hc\pi^2\varepsilon_\circ^2\hbar^2}=1.09\times10^7m^{-1}$ we get:

$$\lambda=\frac{225}{16(1.09\times10^7m^{-1})}=\mathbf{1.29\ \mu m}$$

Problem 6.- A Rydberg atom is a hydrogen atom that is excited close to ionization, for example with n=300. Calculate the frequency of a transition from n=301 to n=300.

Solution: Let us find the wavelength first:

$$\frac{1}{\lambda}=\frac{1}{91.14nm}\left(\frac{1}{300^2}-\frac{1}{301^2}\right)=\frac{1}{91.14nm}\left(\frac{301^2-300^2}{301^2\times300^2}\right)=\frac{1}{91.14nm}\left(\frac{601}{301^2\times300^2}\right)$$

$$\lambda=91.14nm\left(\frac{300^2\times301^2}{601}\right)=1.23\text{ m}$$

Given this wavelength, the frequency is:

$$f=\frac{c}{\lambda}=\frac{3\times10^8m/s}{1.23m}=\mathbf{243\ MHz}$$

Problem 7.- What is the speed of the electrons that are the closest to the nucleus in the iron atom according to the Bohr model? What is the relativistic gamma for this speed?

Solution: The speed can be calculated with:

$$v=\frac{Zc}{137}=\frac{26c}{137}=\frac{26(3\times10^8m/s)}{137}=\mathbf{5.69\times10^7\ m/s}$$

Gamma for this speed is $\gamma=\frac{1}{\sqrt{1-\beta^2}}=\frac{1}{\sqrt{1-(26/137)^2}}=\mathbf{1.0185}$

Problem 8.- For mass spectrometry applications you want a hydrogen atom whose electron has a very low kinetic energy of the order of 0.001eV. Calculate the minimum quantum number ***n*** that will give you that slow electron.

Solution: The kinetic energy of an electron in a hydrogen atom is given by: $K.E. = \frac{13.6eV}{n^2}$, so for the energy to be less than 0.001eV we need:

$$\frac{13.6eV}{n^2} < 0.001eV \rightarrow n > \sqrt{\frac{13.6}{0.001}} = 116.6$$, then n has to be at least **117.**

Problem 9.- In the n=3 state of hydrogen, find the velocity of the electron, its kinetic energy and potential energy.

Solution: According to the Bohr model.

$$v = \frac{Z}{n}\sqrt{\frac{2E_o}{m}} = \frac{1}{3}\sqrt{\frac{2(13.6eV)}{0.511\times10^6 eV/c^2}} = \frac{c}{411} = \mathbf{7.3\times10^5\ m/s}$$

Problem 9a.- In the n=3 state of Be^{+++}, find the velocity of the electron using the Bohr model.

Problem 10.- Calculate the second ionization potential of helium (the energy to remove the electron from a He^+ ion).

Solution: The He^+ ion is a hydrogenic atom and so the Bohr model is appropriate to find the second ionization potential:

$$IP = Z^2 E_o = 2^2(13.6eV) = \mathbf{54.4\ eV}$$

Problem 11.- Mention 3 problems with the Bohr model.

Solution:
a) It gives a sharp value of the orbit radius of the electron, which is not true.
b) The ground state has orbital angular momentum zero, not $\hbar$ like the model predicts.
c) The model is only appropriate for hydrogen or hydrogenic atoms.
d) It predicts a flat atom, and it cannot account for 3 dimensions.

Problem 12.-

a) What is the smallest value of the quantum number *l* if the angular momentum of the electron is less than 1° off the z-axis?

b) What is the degeneracy of the n=7 shell of the hydrogen atom, ignoring magnetic fields.

Solution:

a) The minimum angle will happen when $L_z = l\hbar$ and then:

$$\cos\theta = \frac{l}{\sqrt{l(l+1)}} \rightarrow \cos^2\theta = \frac{l}{l+1} \rightarrow l = \frac{\cos^2\theta}{1-\cos^2\theta} = \frac{\cos^2 1^\circ}{1-\cos^2 1^\circ} = \mathbf{3283}$$

b) You can find that the degeneracy is $2n^2$, so for n=7 it will be **98**.

Problem 13.- To what state "n" do you need to excite hydrogen if you want a Rydberg atom with a radius of at least 1 micron?

Solution: Since the radius of the orbit is $r = n^2 a_0$, to get 1 micron, we need:

$$r = n^2 a_0 = 1\times10^{-6}\,\mathrm{m} \rightarrow n = \sqrt{\frac{1\times10^{-6}\,\mathrm{m}}{a_0}} = \sqrt{\frac{1\times10^{-6}\,\mathrm{m}}{0.529\times10^{-10}\,\mathrm{m}}} = 137.5$$

So, the principal quantum number needs to be **138** at least.