Modern Physics

Notes on Special Relativity

Relativity is not necessary for Newton's second law of motion. The equation is just fine with Galilean transformations that use a universal time for all frames of reference. Relativity is not necessary for Schrödinger's equation either (in its original 1927 form).

Maxwell's equations of electromagnetism predict the possibility of electric fields and magnetic fields that exist without charges in vacuum. The fields would need to travel at a speed = $\frac{1}{\sqrt{\varepsilon_0 \mu_0}}$ which is the speed of light. But this speed should be the same for all frames of reference, so it is incompatible with Galilean transformations.

There was experimental evidence from the Michelson-Morley experiment that the speed of light is the same in all frames of reference, although Einstein was unaware of this.

In 1905 Einstein published the paper "On the Electrodynamics of Moving Bodies" which uses Lorentz transformations instead of Galilean solving the problem of constant speed of light. The same year he published his most famous paper about the equivalence of mass and energy $E=mc^2$

The numerical part of the theory can be described with two numbers that dictate the changes:

$$\beta = \frac{v}{c} \qquad \text{and} \quad \gamma = \frac{1}{\sqrt{1-\beta^2}}$$

These are numbers without units. β must be less than 1 because massive objects cannot reach the speed of light, while γ needs to be 1 or greater.

Consequences

Length contraction: The proper length of an object ($L_\circ$) can only be measured in a frame of reference at rest with respect to the object. If the object is moving, the length of the object appears to be shorter in the direction of motion.

$$L = \frac{L_0}{\gamma}$$

Time dilation: The proper time ($T_\circ$) of how a person ages or how an object decays can only be measured in a frame of reference at rest with respect to the person or object. If the object is moving, the time will appear dilated according to the equation:

$$T = \gamma T_0$$

Linear Momentum: According to Newton, momentum is $p = mv$, but Einstein found that the correct equation should be $p = \gamma mv$ Some people interpret this as if the mass of a moving object increases with its speed. A better interpretation is that the inertia increases, making accelerating the object more difficult, so it appears to be more massive. As a corollary of this, Newton's second law needs to be modified to $F = \gamma ma$ for forces at right angles with the velocity and $F = \gamma^3 ma$ for forces collinear with the velocity.

Galilean relativity (a.k.a. Newtonian relativity) allows you to transform the laws of physics from one inertial frame of reference to another. It basically states that positions can be converted from a frame K to another frame K' using the equations:

$$x' = x - vt$$
$$y' = y$$
$$z' = z$$
$$t' = t$$

Where the frame K' is moving towards the right along the x-axis with velocity v and the two origins coincide at time t'=t=0. This has the consequence that velocities are transformed as follows:

$$v'_x = v_x - v$$
$$v'_y = v_y$$
$$v'_z = v_z$$

These equations work very well for slow speeds and they have been confirmed in everyday life many times. When you apply these equations to Newtonian dynamics, the forces and accelerations are transformed identically. In other words, Newton's second law is invariant under Galilean transformations.

Schrödinger's equation that describes quantum mechanics is also invariant under Galilean transformations. You just need to change variables and the equation is the same, so why do we need special relativity?

Problem with Maxwell's equations: According to Maxwell, electric and magnetic fields in vacuum satisfy the equations:

$$\nabla . D = 0 \qquad \nabla . B = 0 \qquad \nabla \times H = \frac{\partial D}{\partial t} \qquad \nabla \times E = -\frac{\partial B}{\partial t}$$

If you take the curl of the fourth equation, you get:

$$\nabla \times \nabla \times E = -\nabla \times \frac{\partial B}{\partial t} \rightarrow \nabla^2 E = -\varepsilon_o \mu_o \frac{\partial^2 E}{\partial t^2}$$

Where we use the fact that in vacuum: $D = \varepsilon_0 E$ and $B = \mu_0 H$

The last equation tells us that a possible solution to the electric field in vacuum is the trivial $E = 0$, but there are also non-trivial solutions where the electric field exists without charges and travels with speed: $c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}}$ = 3.00×10^8 m/s, which is the speed of light.

This speed should be constant, but according to Galilean relativity when you transform between two different frames moving along the x-axis, the speed along the x-axis will pick up an additional velocity v.

Here you have two possibilities: Either Maxwell's equations are valid only in *one* frame of reference and you use Galilean transformations in all the other frames to calculate the speed of light **or** the speed of light is the same in all frames of reference and Galilean transformations need to be modified.

The Michelson-Morley experiment demonstrated that the speed of light was independent of the frame of reference and Einstein worked out the consequences of using Lorentz transformations instead of Galilean. It is remarkable that Einstein based his theory on the conviction that the speed of light was constant without being aware of the experiment.

Lorentz Transformations: Sometimes you find textbooks that *derive* these transformations. The usual procedure is to assume a linear relation (where x', y', z' and t' are linearly dependent on x, y, z, and t) and find a combination that preserves the speed of light after the transformation. Instead, we are going to state the transformations and derive its consequences:

$$x' = \gamma(x - \beta ct)$$
$$y' = y$$
$$z' = z$$
$$t' = \gamma(t - \beta \frac{x}{c})$$

You can also use the reverse transformation:

$$x = \gamma(x' + \beta ct')$$
$$y = y'$$
$$z = z'$$
$$t = \gamma(t' + \beta \frac{x'}{c})$$

where $\beta = \frac{v}{c}$ and $\gamma = \frac{1}{\sqrt{1 - v^2 / c^2}}$

The first important consequence is that the speed of light is constant in any frame of reference. We can prove this by looking at the wave emitted by a candle in the positive x direction. We find that when you compute x'/t' you also get c.

The other two immediate consequences are that length is contracted when an object is in motion and that time is dilated for clocks that are moving.

Length contraction:
We consider an object at rest in frame K, lying between the origin and the point x=L_o, y=0, z=0. In this frame of reference, it doesn't matter *when* you measure the length, you will always get L_o, which is called the proper length. But in a moving frame of reference K', the measurement has to be done at the same time, so $t'_A = t'_B$, where A and B indicate the events of measuring the position of the two ends of the object.

According to the Lorentz transformations:

$$x'_A = \gamma(x_A - \beta c t_A) = 0 \qquad x'_B = \gamma(x_B - \beta c t_B) = \gamma(L_o - \beta c t_B)$$

$$t'_A = \gamma(t_A - \beta \frac{x_A}{c}) = 0 \quad \text{and} \quad t'_B = \gamma(t_B - \beta \frac{x_B}{c}) = 0 \rightarrow t_B = \beta \frac{x_B}{c} = \beta \frac{L_0}{c}$$

Replacing the second equation in the first and using a bit of algebra gives us:

$$L = \frac{L_0}{\gamma}$$

Note: *remember that the lengths in the other two directions are not contracted!*

Time dilation: If a period of time passes in the same place in a frame of reference it is called the proper time and represented as T_o. For example: aging in the frame of reference of the person that is aging or nuclear decay in the frame of reference of the nucleus, etc. Then, the time is going to be dilated in any other frame of reference where the nucleus or the person is moving. The time in that other frame of reference is going to be:

$$T = \gamma T_0$$

Addition of velocities:

$$u'_x = \frac{u_x - v}{1 - \frac{v u_x}{c^2}} \qquad u_x = \frac{u'_x + v}{1 + \frac{v u'_x}{c^2}}$$

$$u'_y = \frac{u_y}{\gamma\left(1 - \frac{v u_x}{c^2}\right)} \qquad u_y = \frac{u'_y}{\gamma\left(1 + \frac{v u'_x}{c^2}\right)}$$

$$u'_z = \frac{u_z}{\gamma\left(1 - \frac{v u_x}{c^2}\right)} \qquad u_z = \frac{u'_z}{\gamma\left(1 + \frac{v u'_x}{c^2}\right)}$$

Linear Momentum: $\vec{p} = \frac{m\vec{u}}{\sqrt{1 - \frac{u^2}{c^2}}}$

Doppler Effect:

$f_{detected} = f_{source}\sqrt{\frac{1-\beta}{1+\beta}}$ for a receding source (red shift)

$f_{detected} = f_{source}\sqrt{\frac{1+\beta}{1-\beta}}$ for an approaching source (blue shift)

$f_{detected} = f_{source}\sqrt{1-\beta^2}$ for a source moving at right angles (transverse red shift).

Kinetic energy: $\text{K.E.} = (\gamma - 1)\text{mc}^2$

Rest energy: $\text{R.E.} = \text{mc}^2$

Total energy: $\text{E} = \text{K.E.} + \text{R.E.} = \gamma\text{mc}^2$

Ancillary equation: $\text{E}^2 = \text{p}^2\text{c}^2 + \text{m}^2\text{c}^4$

Spacetime interval: It is invariant. It has the same value in any frame of reference.

$\text{s}^2 = x^2 + y^2 + z^2 - c^2t^2$

$\Delta\text{s}^2 = (x_A - x_B)^2 + (y_A - y_B)^2 + (z_A - z_B)^2 - c^2(t_A - t_B)^2$

Rest energy of an electron: $mc^2 = 0.511 MeV$