Modern Physics

Bohr Model

We could consider a hydrogen atom, but to make it more general, let us consider a nucleus with charge *Ze* and a single electron. This is sometimes called a "hydrogenic" atom.

The model starts by assuming that the angular momentum is quantized in multiples of $\hbar$:

$$L = n\hbar$$

Where *n* is a positive integer. This assumption has consequences for the energy, the radius of the orbit, and other quantities, as we will see:

Since $L = mvr$ for a circular orbit, we get: $mvr = n\hbar$, but we also know that the coulomb force is the centripetal force, so:

$$F = \frac{1}{4\pi\varepsilon_\circ}\frac{Ze^2}{r^2} = m\frac{v^2}{r},$$

We can replace $v = \frac{n\hbar}{mr}$ in the equation above and obtain:

$$\frac{1}{4\pi\varepsilon_\circ}\frac{Ze^2}{r^2} = m\frac{\left(\frac{n\hbar}{mr}\right)^2}{r} \rightarrow \boxed{r = \frac{n^2}{Z}\frac{4\pi\varepsilon_\circ\hbar^2}{me^2}}$$

So, the radius of the orbit is proportional to the principal quantum number squared and inversely proportional to the charge of the nucleus. This result defines also:

$$\boxed{a = \frac{4\pi\varepsilon_\circ\hbar^2}{me^2} = 5.3\times10^{-11}\,\mathrm{m}}$$

Which is called the Bohr radius.

The electrostatic potential energy is now easy to calculate:

$$P.E. = -\frac{1}{4\pi\varepsilon_\circ}\frac{Ze^2}{r} = -\frac{1}{n^2}\frac{mZ^2e^4}{16\pi^2\varepsilon_\circ^2\hbar^2}$$

The kinetic energy is half this amount, but positive

$$K.E. = \frac{1}{2}\frac{1}{4\pi\varepsilon_\circ}\frac{Ze^2}{r} = \frac{1}{n^2}\frac{mZ^2e^4}{32\pi^2\varepsilon_\circ^2\hbar^2}$$

As consequence, the total energy is negative and equal to:

$$\boxed{E = -\frac{Z^2}{n^2}\frac{me^4}{32\pi^2\varepsilon_\circ^2\hbar^2}}$$

Where the quantity $E_o = \frac{me^4}{32\pi^2\varepsilon_\circ^2\hbar^2} = 2.17\times10^{-18}J = 13.6eV$

The speed of the electron determined from $\frac{1}{2}mv^2 = \frac{Z^2}{n^2}\frac{me^4}{32\pi^2\varepsilon_\circ^2\hbar^2}$

$$\boxed{v = \frac{Z}{n}\frac{e^2}{4\pi\varepsilon_\circ\hbar}},$$

Therefore, the speed is slower for larger quantum numbers and faster for larger nuclear charge. Notice that here we used the non-relativistic version of the kinetic energy.

The quantity: $v_\circ = \frac{e^2}{4\pi\varepsilon_\circ\hbar} = 2.18\times10^6 m/s$ is the speed of the electron in the ground state.

Also, notice:

$$\alpha = \frac{v_\circ}{c} = \frac{e^2}{4\pi\varepsilon_\circ\hbar c} = \frac{2.18\times10^6 m/s}{3.00\times10^8 m/s} \approx \frac{1}{137}$$

Which is the fine structure constant.