Optics

Blackbody radiation

Stefan-Boltzmann radiation law $\frac{Power}{Area} = \sigma T^4$ where $\sigma = 5.67\times10^{-8}\frac{W}{m^2K^4}$

Wien's law $\lambda_{MAX}T = 2.897\times10^{-3}mK$ (meter kelvin, not millikelvin)

Problem 1.- A star of radius R_o and surface temperature T_o emits a total amount of power P_o. Another star has a radius $3R_o$ and surface temperature $2T_o$. Calculate the total power emitted by the second star.

(A) 2 P_o
(B) 3 P_o
(C) 16 P_o
(D) 48 P_o
(E) 144 P_o

Solution: Using Stefan-Boltzmann law $Power = Area\times\sigma T^4 = 4\pi\sigma R^2T^4$

So, the power of the second star will be $3^2\times2^4$ times the power of the first one.

Answer: **E**

Problem 2.- The surface of the Sun has a temperature close to 6,000 K and it emits a blackbody spectrum that reaches a maximum near 500 nm.
For a super-hot star whose surface temperature is 12,000 K, at what wavelength would the thermal spectrum reach a maximum?

(A) 16 μm
(B) 4 μm
(C) 1 μm
(D) 250 nm
(E) 125 nm

Solution: If the temperature is twice as much, the wavelength at maximum intensity will be half of 500nm or 250nm

Answer: **D**

Problem 3.- Consider a sphere with constant surface temperature in equilibrium with the radiation emitted by the sun. Calculate the temperature of the surface for the main distance of the planets to the sun.

Present your data in a table including all 8 planets and Pluto.

Indicate in your table which planets would have a temperature below liquid nitrogen boiling temperature at 1 atmosphere (77.3K), liquid helium (4.2K), liquid hydrogen (20.3K) and liquid oxygen (90.2)

Solution:

	Planet	Mean distance to sun	Intensity	T
1	Mercury	0.3871	9009.22	446.4
2	Venus	0.7233	2580.46	326.6
3	Earth	1.0000	1350	277.8
4	Mars	1.5240	581.251	225.0
5	Jupiter	5.2030	49.8685	121.8
6	Saturn	9.5370	14.8426	89.9
7	Uranus	19.1913	3.66543	63.4
8	Neptune	30.0690	1.49312	50.7
9	Pluto	39.4820	0.86604	44.2

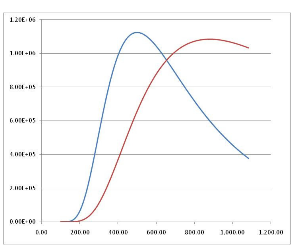
We find that the average temperature of Saturn is below liquid oxygen and Uranus is below liquid nitrogen. Liquid hydrogen and helium temperatures are not reached in any of the planets (considering average temperatures, that is).

Problem 4.- Use a spreadsheet program to plot the energy density as a function of frequency and wavelength for the temperature of the surface of the sun (T=5800K).
Indicate at what frequency and at what wavelength is the maximum density.

$$U(f) = \frac{8\pi h f^3}{c^3\left[\exp\left(\frac{hf}{k_B T}\right) - 1\right]}$$

$$U(\lambda) = \frac{8\pi hc}{\lambda^5\left[\exp\left(\frac{hc}{k_B T\lambda}\right) - 1\right]}$$

Solution: The energy density in terms of wavelength is plotted as the blue line in the figure below:



The maximum happens when $\lambda = 499nm$, which is also the value obtained used Wien's law.

The density as a function of frequency is a different function, which is depicted as the red line in the figure above after multiplying it by 7×10^{20} (otherwise, it would not be visible on that scale). Notice that the maximum of this function falls in the IR region at $\lambda = 832nm$ or frequency of $f = 3.6\times10^{14}Hz$

Problem 5.- What would be the temperature of a filament in a light bulb if you wanted the maximum intensity to be at 600nm?

Solution: Using Wien's law:

$$\lambda T = 0.0029mK \rightarrow T = \frac{2.9\times10^{-3}}{600\times10^{-9}} = \mathbf{4{,}830\ K}$$

Problem 6.- A blackbody emits 10mW when its temperature is T_1. Calculate the power emitted if the temperature is 2 T_1

(A) 5 mW
(B) 10 mW
(C) 20 mW
(D) 100 mW
(E) 160 mW

Solution: Since the power depends on T^4 when doubling the temperature the intensity increases by a factor of 16.
Answer: **(E) 160 mW**

Problem 6a.- The energy from electromagnetic waves in equilibrium in a cavity is used to melt wax. If the absolute temperature of the cavity is increased by a factor of three, the mass of wax that can be melted in a fixed amount of time is increased by a factor of

(A) 3
(B) 6
(C) 9
(D) 27
(E) 81

Solution: Since the power depends on T^4
Answer: **(E) 81**

Problem 7.- The wavelength at which the density of electromagnetic radiation $U(\lambda)$ reaches a maximum is given by the well-known equation

$$\lambda T = 2.897mmK$$

This is sometimes called Wien's law.

Find a similar equation for the frequency U(f). You can follow this procedure:

-Take the derivative of U(f) with respect to f.
- Make that derivative equal to zero
- Solve for f (or f/T)
- Approximate the exponential to a very large number (consider $\exp \to \infty$)

Solution: We take the derivative of U(f) with respect to f:

$$U(f) = \frac{8\pi h f^3}{c^3\left[\exp\left(\frac{hf}{k_B T}\right) - 1\right]}$$

$$\to \frac{dU(f)}{df} = \frac{8\pi h\left[3f^2\left[\exp\left(\frac{hf}{k_B T}\right) - 1\right] - f^3\left(\frac{h}{k_B T}\right)\exp\left(\frac{hf}{k_B T}\right)\right]}{c^3\left[\exp\left(\frac{hf}{k_B T}\right) - 1\right]^2}$$

Make that derivative equal to zero

$$3f^2\left[\exp\left(\frac{hf}{k_B T}\right) - 1\right] - f^3\left(\frac{h}{k_B T}\right)\exp\left(\frac{hf}{k_B T}\right) = 0$$

Solve for f (or f/T)

$$\frac{f}{T} = \frac{3k_B\left[\exp\left(\frac{hf}{k_B T}\right) - 1\right]}{h\exp\left(\frac{hf}{k_B T}\right)}$$

Since the exponential is typically very large we can make the approximation:

$$\frac{f}{T} \approx \frac{3k_B}{h} = \mathbf{6.25 \times 10^{10}\ Hz/K}$$

This equation will give results for wavelengths that are larger than Wien's law by a factor of 5/3.

Problem 8.- The density of electromagnetic energy as a function of wavelength is given by the equation:

$$U(\lambda) = \frac{8\pi hc}{\lambda^5\left[\exp\left(\frac{hc}{k_B T\lambda}\right) - 1\right]}$$

In what units is $U(\lambda)$?

Solution: It is density of energy (J/m^3) per wavelength (m), so **J/m^4**

You can also do a unit analysis as follows

$$[U(\lambda)]:\left[\frac{8\pi hc}{\lambda^5\left[\exp\left(\frac{hc}{k_BT\lambda}\right)-1\right]}\right]:\frac{[8\pi hc]}{\left[\lambda^5\left[\exp\left(\frac{hc}{k_BT\lambda}\right)-1\right]\right]}:\frac{[8][\pi][h][c]}{[\lambda]^5\left[\exp\left(\frac{hc}{k_BT\lambda}\right)-1\right]}$$

Notice that the exponential in the denominator is just a number without units, then

$$[U(\lambda)]:\frac{Jsm/s}{m^5}=\frac{J}{m^4}$$