Optics

Planck's radiation

Consider a cavity of length L with metallic walls. Since metals are good conductors of electricity, the electric field at each wall will have to be zero. Then the only stationary modes allowed are:

$\lambda = \frac{2L}{n}$, where n is a positive integer.

That mode will have an energy equal to $E_n = Nhf = N\frac{hc}{\lambda} = N\frac{hcn}{2L}$, where N is any nonnegative integer.

According to statistical physics the average energy stored in this mode will be:

$$\langle E_n \rangle = \frac{\sum_{N=0}^{\infty} N\frac{hcn}{2L}\exp\left(-N\frac{hcn}{2Lk_BT}\right)}{\sum_{N=0}^{\infty}\exp\left(-N\frac{hcn}{2Lk_BT}\right)} = \frac{hcn}{2L\left[\exp\left(\frac{hcn}{2Lk_BT}\right)-1\right]}$$

If we wanted to calculate the density of energy we would need to add the contribution from each mode and divide by the volume of the cavity. Let's assume the cavity is a cube, so there will be a contribution from each possible direction and each polarization (a factor of 2).

The modes will be combinations of three different orientations: (n_x, n_y, n_z) with an energy:

$$E_{nx,ny,nz} = \frac{2hcn}{2L\left[\exp\left(\frac{hcn}{2Lk_BT}\right)-1\right]}, \text{ where } n = \sqrt{n_x^2 + n_y^2 + n_z^2}$$

Now to find out the number of modes in a wavelength interval notice that the number of modes grows like 1/8 the volume of a sphere:

$$\#\text{modes } (n < n_o) = \frac{1}{8}\left(\frac{4}{3}\pi n_o^3\right) \rightarrow \#\text{modes } (\lambda > \lambda_o) = \frac{1}{8}\left(\frac{4}{3}\pi\left[\frac{2L}{\lambda_o}\right]^3\right) = \frac{4}{3}\pi\frac{L^3}{\lambda_o^3}$$

Taking a derivative to find the density of modes:

$\frac{d(\#\text{modes})}{d\lambda} = -4\pi\frac{L^3}{\lambda_o^4}$, where the sign indicates that the larger the wavelength the less the number of modes.

Going back to the energy, we can now calculate the density of energy per wavelength and per volume of the cavity:

$$U(\lambda)=\frac{4\pi L^3}{\lambda^4}\frac{2hcn}{2L\left[\exp\left(\frac{hcn}{2Lk_BT}\right)-1\right]L^3}$$

$$U(\lambda)=\frac{8\pi hc}{\lambda^5\left[\exp\left(\frac{hc}{k_BT\lambda}\right)-1\right]}$$

To get the radiation we multiply by c/4, the speed of light and a factor of four due to the geometry of the vector distribution.

$$R=\frac{2\pi hc^2}{\lambda^5}\frac{1}{e^{\frac{hc}{\lambda k_BT}}-1}$$