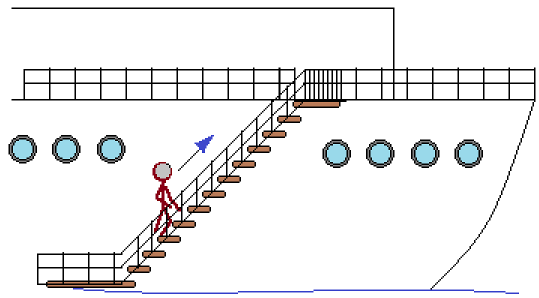
Physics I

More kinematics problems

Problem 1.- A passenger in a cruise ship that travels at 1 m/s in steady waters climb stairs at a speed of 0.5 m/s with respect to the ship. The stairs make an angle of 45º over the horizontal and points in the same direction as the ship motion, as shown below. What is the velocity of the passenger?



Solution: The speed of 0.5m/s is with respect to the ship, so to calculate the velocity we will need to add the two vectors, ship velocity and passenger with respect to ship.
We choose a coordinate system where the X-axis is horizontal, and the Y-axis is vertical. In this system:

$$\vec{v} = (1,0) + (0.5\cos 45^\circ + 0.5\sin 45^\circ)$$

$$\vec{v} = \mathbf{(1.35,\ 0.354)\ m/s}$$

Problem 2.- The velocity of an object in a horizontal plane is given by the function:
$\vec{v} = (-2, 4t)$
The position of the object at t = 0 is $\boldsymbol{r}$ = (1.5, 3).

a) Find the instantaneous position and acceleration.
b) Make a graph of the trajectory and describe it.
c) Calculate the average speed between 0 and 2 seconds.
d) Find the tangential and radial accelerations at t = 1 second.
e) Find the average acceleration and velocity between 0 and 1 second.
f) If a second object has a velocity $\vec{v} = (-2, 4)m/s$ and at t=0 is at the origin of coordinates, what is its trajectory? And, will it collide with the first object?
g) Graph the position, velocity, and acceleration components of each object.

Solution:
a) To find the instantaneous position we add the initial position to the integral of the velocity from 0 to t.

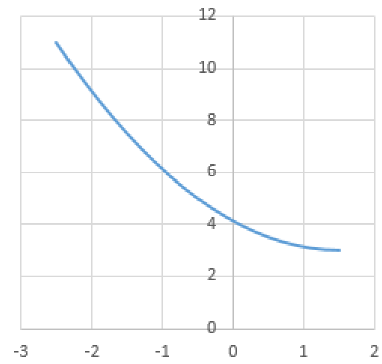
$$\vec{r} = (1.5, 3) + \int_0^t (-2, 4t)dt$$

***r* = (1.5-2t, 3+2t²) m**

The derivative of velocity with respect to time gives us the acceleration:

$$\vec{a} = \frac{d}{dt}(-2, 4t) = \mathbf{(0,\ 4)\ m/s^2}$$

b) Since the acceleration is constant, the trajectory is parabolic. Using the equation found in (a) we can graph it below.



c) We find the distance traveled by integrating the speed between 0 and 2 seconds. By dividing this distance by 2s, we find the average speed in that interval.

$$v = \sqrt{4+16t^2}$$

$$d = \int_0^2 \sqrt{4+16t^2}\,dt$$

$$v_{average} = \frac{d}{t} = \frac{\int_0^2 \sqrt{4+16t^2}\,dt}{2}$$

We can integrate by using the substitution method with $t = 0.5\sinh\phi$

Changing the limits of integration to

$$t = 0 \rightarrow \phi = 0$$

$$t = 2 \rightarrow \phi = \sinh^{-1} 4$$

The integrating function becomes $\sqrt{4+16t^2} = 2\cosh\phi$

The differential is now $dt = 0.5\cosh\phi d\phi$

Then, $d = \int_0^{\sinh^{-1}(4)} \cosh^2\phi d\phi$

$$d = \int_0^{\sinh^{-1}(4)} \frac{\cosh 2\phi + 1}{2} d\phi$$

$$d = \frac{\sinh 2\phi + 2}{4}\Big|_0^{\sinh^{-1}(4)} = \frac{\sinh 2\sinh^{-1} 4 + 2\sinh^{-1} 4}{4} = 9.29\text{m}$$

The average speed is

$$v = \frac{9.29m}{2s} = \mathbf{4.65\ m/s}$$

d) The acceleration is constant, as we found in part (a) $\vec{a} = (0,4)m/s^2$

The projection of this acceleration in the direction of the velocity is the tangential acceleration. Notice that the velocity at time t=1s is

$$\vec{v}(1) = (-2,4)m/s$$

Therefore, the unit vector tangent to the trajectory at that instant is $\hat{u} = \frac{(-2,4)}{\sqrt{4+16}} = \frac{(-1,2)}{\sqrt{5}}$

To find the magnitude of the tangential acceleration we take a dot product between the acceleration with the unit vector.

$$a_T = \vec{a} \cdot \hat{u} = (0,4) \cdot \frac{(-1,2)}{\sqrt{5}} = \frac{8\sqrt{5}}{5} m/s^2$$

The tangential acceleration vector is its magnitude times the unit vector:

$$\vec{a}_T = (\vec{a} \cdot \hat{u})\hat{u} = \frac{8\sqrt{5}}{5}\frac{(-1,2)}{\sqrt{5}} = \mathbf{(-1.6,\ 3.2)\ m/s^2}$$

To find the radial acceleration we subtract the tangential acceleration from the total acceleration:

$$\vec{a}_R = \vec{a} - \vec{a}_T = (0,4) - (-1.6,3.2) = \mathbf{(1.6,\ 0.8)\ m/s^2}$$

Its magnitude is $a_R = \sqrt{16 - \frac{64}{5}} = \sqrt{\frac{16}{5}} = \frac{4\sqrt{5}}{5} m/s^2$

e) Since the acceleration is constant, its average value is the same

$\vec{a}_{average} =$ **(0, 4) m/s²**

To find the average velocity we can take the displacement divided by time.

$$\vec{v}_{average} = \frac{\vec{r}_{final} - \vec{r}_{initial}}{t_{final} - t_{initial}} = \frac{(1.5 - 2\times 1, 3 + 2\times 1^2) - (1.5, 3)}{1 - 0} =$$ **(-2, 2) m/s**

f) The second object has constant velocity, so its trajectory is a straight line. Its position is:

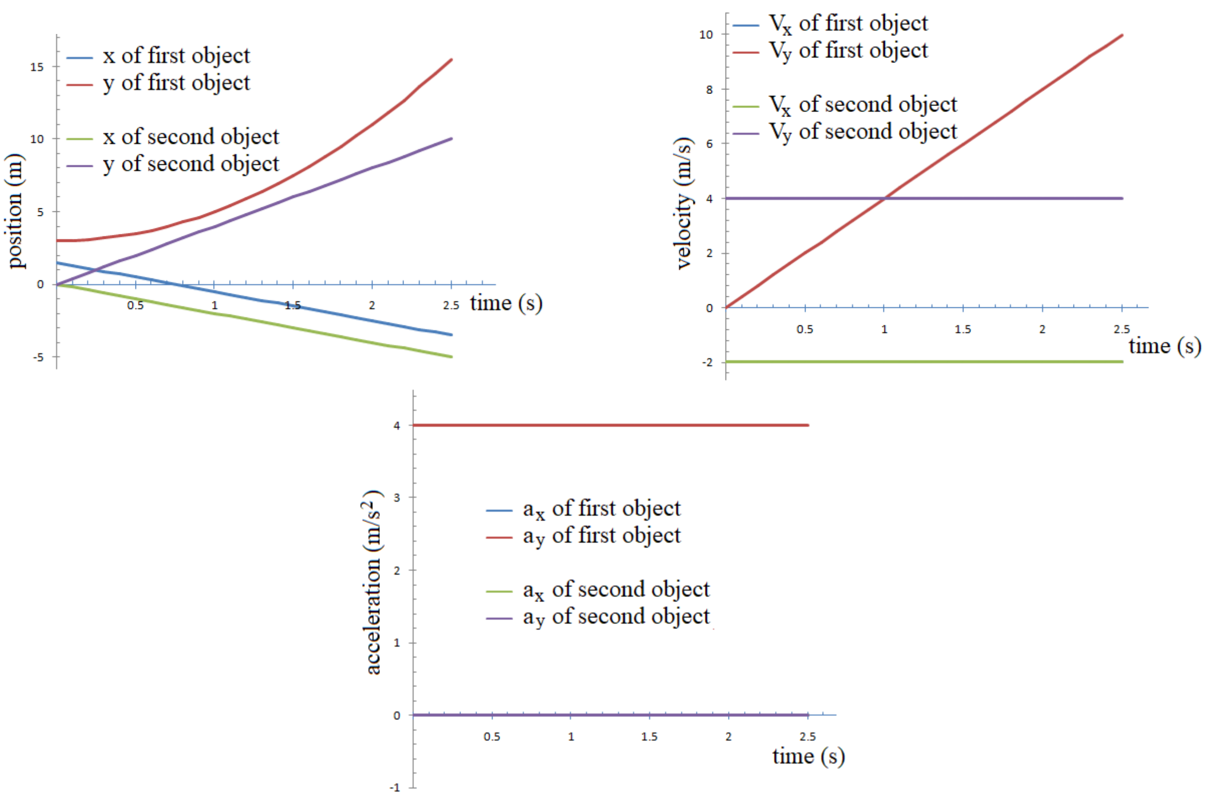
$$\vec{r} = (-2t, 4t)m$$

To find out if it will collide with the first object, they have to have the same coordinates at the same time, which means that:

$$(1.5 - 2t, 3 + 2t^2) = (-2t, 4t)$$

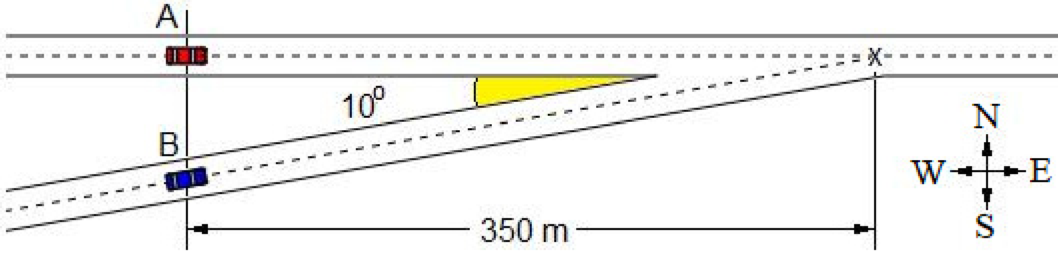
This will never happen, since the coordinate x of the first object will always be larger than the second one.

g) The graphs of position, velocity, and acceleration of the two objects are shown below.



Problem 3.- A car A is traveling along a highway towards the east at a constant velocity 35m/s. Another car B is entering the highway by a ramp pointing 10º north of east at a speed *v*. The point marked X in the figure is 350m from A.

Using a coordinate system x-y for east-north, calculate how the distance between the cars changes over time and find the safe values of *v* that will avoid a collision.



Solution: Taking the origin of coordinates at the initial position of car A, the coordinates of the two cars are:

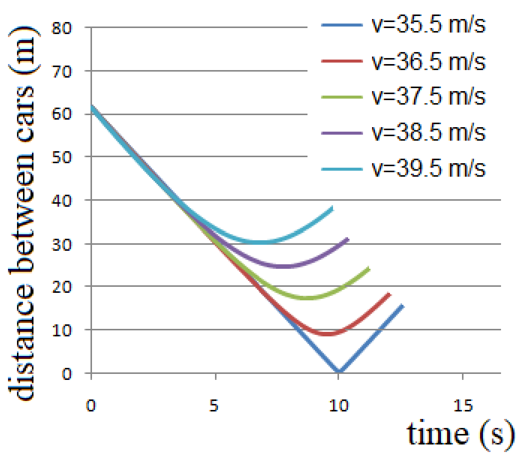
$\vec{r}_A = (35t, 0)m$

$\vec{r}_B = (vt\cos 10^\circ, -350\tan 10^\circ + vt\sin 10^\circ)m$

The distance between them is

$$d = |\vec{r}_A - \vec{r}_B| = \sqrt{(35t - vt\cos 10^\circ)^2 + (350\tan 10^\circ - vt\sin 10^\circ)^2}$$

This function of time depends on the value of *v* and time. This is a family of functions with *v* as a parameter, as shown below.



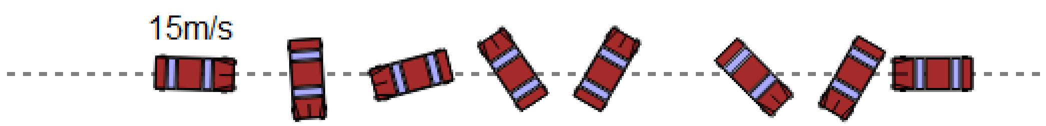
To avoid a collision, we should make sure that they do not reach point X at the same time. Since car A covers the 350m at 35m/s, it will take it 10s to reach X.

On the other hand, the distance for car B to get to point X is $d = \dfrac{350}{\cos 10^\circ}$ and will reach that point in a time $t = \dfrac{350}{v\cos 10^\circ}$

We need to make sure that this time is longer or shorter than 10s (plus a security margin), so car B's velocity must be larger or smaller than

$$10 = \frac{350}{v\cos 10^\circ} \rightarrow v_{collision} = \frac{350}{10\cos 10^\circ} = 35.5 m/s$$

Problem 4.- A car rotates 1.5 revolutions while it slides until it stops. Initially, its center of mass was moving at 15m/s, but due to friction with the ice, its speed reduced at a rate of 1.5m/s^2. Seen from above, the car rotated clockwise. Find its average angular velocity during the slide.



Solution: Since the car's initial velocity was 15m/s and the acceleration 1.5m/s^2, it will take 10s to stop. In that time, it rotated 1.5 revolutions, so its average angular velocity was

$$\omega = \frac{1.5 rev}{10 s} = 0.15 rev/s$$

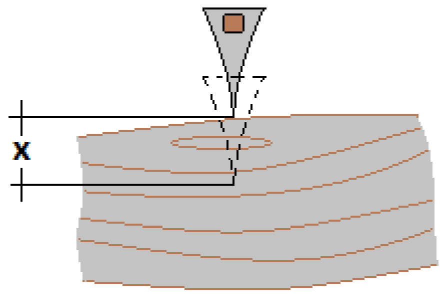
In radians per second this is

$$\omega = 0.15 rev/s\left(\frac{2\pi rad}{1 rev}\right) = 0.3\pi\frac{rad}{s} = 0.942\frac{rad}{s}$$

For a vector representation, we take the vertical upward direction as the z-axis, giving:

$$\vec{\omega} = (0, 0, -0.942)\frac{rad}{s}$$

Problem 5.- An axe hits a log with initial velocity v_0. The deceleration produced by the wood can be described by the function $a = -kx^3$. Calculate how deep the axe will penetrate in the wood.



Solution: According to the problem:

$$a = \frac{dv}{dt} = -kx^3$$

We can re-write the equation as follows:

$$\frac{dv}{dt} = \frac{dv}{dt}\frac{dx}{dx} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx} = -kx^3$$

We can now separate variables and integrate:

$$vdv = -kx^3dx \rightarrow \int_{v_0}^{0} vdv = \int_{0}^{x} -kx^3dx \rightarrow \frac{v_0^{\ 2}}{2} = k\frac{x^4}{4}$$

$$\rightarrow x = \sqrt[4]{\frac{2v_0^{\ 2}}{k}}$$