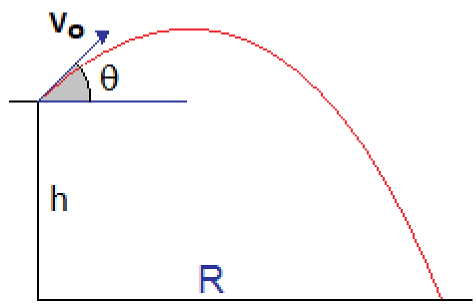
Physics I

Maximum Range, Uneven Terrain

Calculate the angle Θ that gives the maximum range R for a projectile that is shot at an initial velocity v_o from a height h above the ground.



Solution: To solve the problem we notice that in the vertical direction:

The initial velocity is $v_{y1} = v_o \sin\theta$

The final value of y is $y = -h$

We calculate the time with the equation: $y = v_{y1}t - \frac{1}{2}gt^2$

$$\frac{1}{2}gt^2 - v_o t\sin\theta - h = 0 \rightarrow t = \frac{v_o\sin\theta + \sqrt{v_o^2\sin^2\theta + 2gh}}{g}$$

With this time, we now find an equation of the range starting with the equation

$$R = x = v_{1x}t = v_o t\cos\theta$$

Replacing t with the expression calculated above:

$$R = \frac{v_o^2}{g}\left(\sin\theta + \sqrt{\sin^2\theta + \frac{2gh}{v_o^2}}\right)\cos\theta$$

Let us change variable: $z = \frac{2gh}{v_o^2}$

The range is $R = \frac{v_o^2}{g}\left(\sin\theta + \sqrt{\sin^2\theta + z}\right)\cos\theta$

To find the maximum let's take derivative with respect to θ

$$\frac{dR}{d\theta} = \frac{v_o^2}{g}\left[-(\sin\theta + \sqrt{\sin^2\theta + z})\sin\theta + \left(1 + \frac{\sin\theta}{\sqrt{\sin^2\theta + z}}\right)\cos^2\theta\right]$$

This should be zero if there is a maximum, so:

$$(\sin\theta+\sqrt{\sin^2\theta+z})\sin\theta=\left(1+\frac{\sin\theta}{\sqrt{\sin^2\theta+z}}\right)\cos^2\theta$$

$$\left(1+\frac{\sqrt{\sin^2\theta+z}}{\sin\theta}\right)\sin^2\theta=\left(1+\frac{\sin\theta}{\sqrt{\sin^2\theta+z}}\right)\cos^2\theta$$

With some simplifications we get

$$\frac{1-2\sin^2\theta}{\sin^2\theta}=z\rightarrow\sin\theta=\sqrt{\frac{1}{z+2}}$$

The solution is: $\theta=\sin^{-1}\left(\sqrt{\frac{1}{z+2}}\right)=\sin^{-1}\left(\sqrt{\frac{1}{2+\frac{2gh}{v_o{}^2}}}\right)$

Another way of writing the equation is:

$$\theta=\tan^{-1}\left(\frac{v_o}{\sqrt{v_o{}^2+2gh}}\right)$$

Notice that the numerator is the initial speed and the denominator is the final speed (just before hitting the ground).

Consider some cases:

a) h=0, in this case it is the well-known problem of maximum range for level terrain in which case we get 45º.

b) h<0, the denominator will be less than the numerator and the angle will be greater than 45º.

c) If the value of the denominator approaches 0 we will get closer to 90º at which point all the velocity has to be used in reaching the height h and the range will be zero.

d) If h>0, as shown in the figure above, the angle for maximum range will be less than 45º.

For a numerical example with $g=9.8m/s^2$, $v_o=5m/s$ and $h=4m$

$$\theta=\tan^{-1}\left(\frac{5}{\sqrt{5^2+2\times9.8\times4}}\right)=\mathbf{26.2}^\circ$$

You can also prove something interesting: The horizontal velocity is

$$v_{x1} = v_o \cos\left(\tan^{-1}\frac{v_o}{\sqrt{v_o^2 + 2gh}}\right) = v_o \frac{\sqrt{v_o^2 + 2gh}}{\sqrt{2v_o^2 + 2gh}}$$

The angle of the final velocity is:

$$\theta_{final} = -\cos^{-1}\left(\frac{v_{x1}}{\sqrt{v_o^2 + 2gh}}\right) = -\cos^{-1}\left(\frac{v_o}{\sqrt{2v_o^2 + 2gh}}\right) = -\tan^{-1}\left(\frac{\sqrt{v_o^2 + 2gh}}{v_o}\right)$$

Notice that this is the inverse of the tangent of the initial angle.

That means that the initial and final velocities make an angle of 90°. This is clear for the case of even terrain h=0 when the angles are ±45º, but it is also true for maximum range for all values of h.

Maximum range equation for uneven terrain

Now that we have the angle for best range, we can calculate R getting:

$$R = \frac{v_o}{g}\sqrt{v_o^2 + 2gh}$$

We can also look at the situation where R and h are given and we want the minimum speed or kinetic energy to connect the two points with a parabola. We can get this by solving for v_o in the above equation.

$$v_o = \sqrt{g(h + \sqrt{h^2 + R^2})}$$

Alternatively

$$v_o = \sqrt{g(h + D)}$$

D the distance straight between the initial and final points

Let us check two cases:

a) h=0, the minimum velocity will be $v_o = \sqrt{gR}$, which we get with an angle of 45º in the range equation for even terrain.

b) R=0, the minimum velocity to reach height h is $v_o = \sqrt{2gh}$, which we can get with energy considerations or cinematic equations as well.