Physics I

Velocity Calculus

Definition of velocity in one dimension: $v = \frac{dx}{dt}$

Definition of acceleration in one dimension: $a = \frac{dv}{dt} = \frac{d^2x}{dt^2}$

Problem 1.- A satellite in polar orbit moves towards the north at 8,550 m/s when it collides with another satellite in equatorial orbit, which at that point was moving towards the east at 8,250 m/s. Calculate the speed of the second satellite with respect to the first.

Solution: The two vectors can be written as components assuming east to be the positive x-axis and north the positive y-axis:

$$\vec{V}_{satellite\ A} = (0, 8550)$$
$$\vec{V}_{satellite\ B} = (8250, 0)$$

To find the relative velocity we subtract one vector from the other:

$$\vec{V}_{satellite\ B} - \vec{V}_{satellite\ A} = (8250, -8550)$$

The speed is then: $\left|\vec{V}_{satellite\ B} - \vec{V}_{satellite\ A}\right| = \sqrt{8250^2 + 8550^2} =$ **11,900m/s**

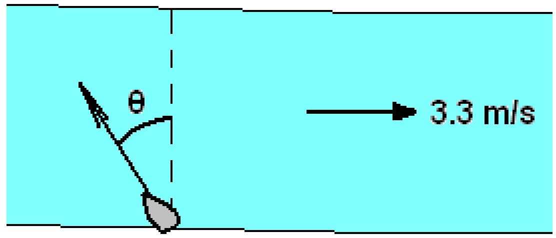
Problem 2.- An oil droplet of mass m is falling in air and experiences a drag force equal to –bv where b is a constant proportional to the viscosity of air.
Calculate its terminal velocity and its kinetic energy when that velocity is reached.
[Ignore buoyancy]

Solution: Terminal velocity will be reached when the net force is zero, which is when the weight and the drag force are equal:

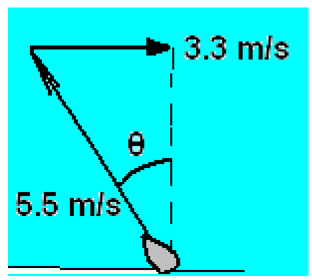
$mg = bv \rightarrow v = \frac{mg}{b}$

At that point the kinetic energy is: $K.E. = \frac{1}{2}mv^2 = \frac{m^3 g^2}{2b^2}$

Problem 3.- A person wants to cross a river with a motorboat that has a speed of 5.5 m/s, however, the water current has a speed of 3.3 m/s. Calculate the angle needed to get straight to the other side.



Solution: The addition of the boat velocity plus the water velocity should give a velocity directly across the river, so the vectors will form a right triangle:



The angle can be calculated from trigonometry:

$$\sin\theta = \frac{3.3m/s}{5.5m/s} = 0.6 \rightarrow \theta = \sin^{-1}(0.6) = \mathbf{37^o}$$

Problem 4.- A projectile with initial velocity v_o enters a viscous fluid where the acceleration is given by: $a = -kv^2$, find the velocity as a function of time.

Useful integral: $\int \frac{dx}{x^2} = -\frac{1}{x} + C$

Solution: The acceleration is defined as the derivative of the velocity with respect to time, so:

$$\frac{dv}{dt} = -kv^2$$

To solve the problem, we separate variables: Velocity on one side and time on the other:

$$\frac{dv}{v^2} = -kdt$$

And now we integrate:

$$-\frac{1}{v}+C=-kt \rightarrow v=\frac{1}{C+kt}$$

The initial condition allows us to find the value of C:

$$v_o=\frac{1}{C+k(0)} \rightarrow C=\frac{1}{v_o}$$

So, the velocity is given by:

$$v=\frac{v_o}{1+kv_o t}$$

Problem 5.- An object dropped on Titan (Saturn's largest moon) is attracted to the surface with an acceleration equal to: $a=g_{\text{Titan}}-kv$

Where $g_{\text{Titan}}=1.35\frac{m}{s^2}$ is the acceleration due to gravity and $k=0.033s^{-1}$ is a resistance due to the viscosity of Titan's dense atmosphere.

Find the terminal velocity of the object.

Solution: If terminal velocity is reached the acceleration is zero, so:

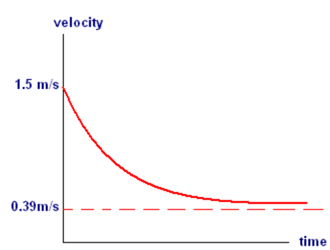
$$0=g_{\text{Titan}}-kv$$

So, the velocity is: $v=\frac{g_{Titan}}{k}=\frac{1.35m/s^2}{0.033/s}=$ **41 m/s**

Problem 6.- A small object falls with initial velocity v_o=1.5m/s in a viscous fluid where the acceleration is given by: $a=g-25v$, find the terminal velocity and sketch the velocity as a function of time.

Solution: The terminal velocity will be reached when the acceleration is zero, so:

$$a=g-25v_{\text{terminal}}=0 \rightarrow v_{\text{terminal}}=\frac{g}{25}=0.392m/s$$



Problem 7.- If the position of a 2.5 kg particle is described by the vector:

$\vec{r} = (t, 5\sin t)$

Find the net force acting on the particle.

Solution: We find the acceleration:

$$\vec{r} = (t, 5\sin t)$$
$$\vec{v} = \frac{d\vec{r}}{dt} = (1, 5\cos t)$$
$$\vec{a} = \frac{d\vec{v}}{dt} = (0, -5\sin t)$$

And the force is:

$$\vec{F} = m\vec{a} = (0, -12.5\sin t)$$

Problem 8.- A particle moves in a straight line following the equation:

$$x = 8t^2 + 5t + 1$$

Determine the position, velocity, and acceleration at t = 2.0 s.

Solution:

a) $x = 8t^2 + 5t + 1 = 8\times 2^2 + 5\times 2 + 1 = \mathbf{43}$

b) $v = \frac{dx}{dt} = 16t + 5 = 16\times 2 + 5 = \mathbf{37}$

c) $a = \frac{dv}{dt} = \mathbf{16}$

Problem 9.- A particle follows a trajectory described by the equation:

$x(t) = 25t^3 + 5t^2 + 20$

Where t is the time in seconds, find the velocity and acceleration at t=2 seconds.

Solution:

$$v(t) = \frac{dx}{dt} = 75t^2 + 10t \rightarrow v(2) = \mathbf{320\ m/s}$$

$$a = \frac{dv}{dt} = 150t + 10 \rightarrow a(2) = \mathbf{310\ m/s^2}$$

Problem 10.- A particle has a velocity described by the equation:

$$\mathrm{v(t) = 5t^3}$$

Where t is the time in seconds, find the displacement from $t_1 = 1$ to $t_2 = 5$ seconds

Solution:

$$x = \int v dt = \int_1^5 5t^3 dt = \left.\frac{5t^4}{4}\right|_1^5 = \frac{5\times 5^4}{4} - \frac{5\times 1^4}{4} = \mathbf{780\ m}$$

Problem 11.- A particle follows a trajectory described by the equation:

$$x(t) = 1.5t^3 + 0.5t + 1$$

Where t is the time in seconds, find the velocity and acceleration as a function of time.

Solution:

$$x(t) = 1.5t^3 + 0.5t + 1$$
$$v(t) = 4.5t^2 + 0.5$$
$$a(t) = 9t$$

Problem 12.- The acceleration of a falling object in a viscous fluid is given by $a=Ae^{-bt}$, calculate the velocity as a function of time if the initial velocity is V_o

Solution: By definition $a = \frac{dv}{dt}$, so: $\frac{dv}{dt} = Ae^{-bt}$ and integrating:

$$\int dv = \int Ae^{-bt} dt \rightarrow v = \frac{Ae^{-bt}}{-b} + C$$

Now we use the initial condition to find the constant:

$$v_o = \frac{Ae^{-b(0)}}{-b} + C \rightarrow C = v_o + \frac{A}{b}$$, so the solution is:

$$v = \frac{Ae^{-bt}}{-b} + v_o + \frac{A}{b}$$
$$v = v_o + \frac{A}{b}\left(1 - e^{-bt}\right)$$

Problem 13.- A particle follows a trajectory described by the equation $x = 5 + 8t - t^2$

a) Calculate the instantaneous velocity of the particle.
b) Find at what time the velocity is zero.
c) Using the time calculated in (b) calculate the maximum value of *x*.

Solution:

a) $v = \dfrac{dx}{dt} = 8 - 2t$

b) $v = 0 \rightarrow 8 - 2t = 0 \rightarrow t = 4$

c) $x\big|_{t=4} = 5 + 8\times 4 - 4^2 = 21$

Problem 14.- The position of a particle is given by x=10t^3+3, y=5t^2-14t, where x and y are in meters and t is in seconds. Find the instantaneous acceleration of the particle at t=2 seconds.

Solution: To find the acceleration we derive twice:

$$a_x = \frac{d^2x}{dt^2} = \frac{d^2}{dt^2}\left(10t^3 + 3\right) = 60t = \mathbf{120\ m/s^2}$$

$$a_y = \frac{d^2y}{dt^2} = \frac{d^2}{dt^2}\left(5t^2 - 14t\right) = \mathbf{10\ m/s^2}$$

Problem 15.- The position of a particle is given by x=8t+3, y=9t^2-14t. Find the average velocity of the particle between t=2s and t=5s.

Solution: We find the displacement vector and divide by the time to get the average velocity:

$$\Delta x = x(5) - x(2) = 8\times 5 + 3 - (8\times 2 + 3) = 24$$

$$\Delta y = y(5) - y(2) = 9\times 5^2 - 14\times 5 - (9\times 2^2 - 14\times 2) = 147$$

$$\vec{V} = \left(\frac{24}{3}, \frac{147}{3}\right) = \mathbf{(8,\ 49)}$$

Problem 16.- A particle follows a trajectory described by the equation:

$$x(t) = \frac{A}{t^2 + b}$$

Find the velocity as a function of time.

Solution: Using the definition of instantaneous velocity:

$$v = \frac{dx}{dt} = \frac{d}{dt}\frac{A}{t^2 + b} = -\frac{2At}{(t^2 + b)^2}$$