Physics Courseware

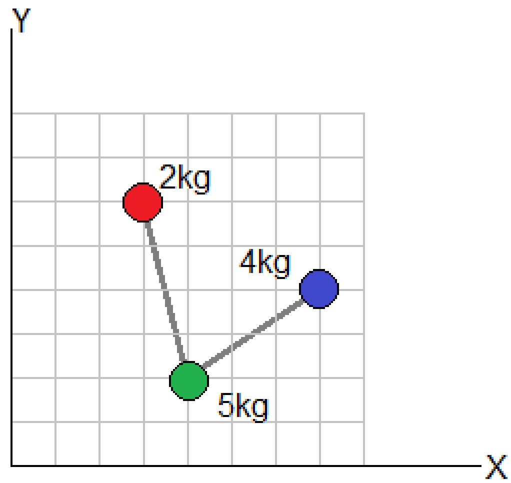
Physics I

Center of Mass

Center of mass: $X_{CM} = \frac{\sum m_i x_i}{\sum m_i}$ and $Y_{CM} = \frac{\sum m_i y_i}{\sum m_i}$

Or $X_{CM} = \frac{\int x dm}{\int m}$ and $Y_{CM} = \frac{\int y dm}{\int m}$

Problem 1.- Find the center of mass X_{CM} and Y_{CM} of the following object made with three masses connected with thin rods.
Ignore the mass of the rods and take each square in the grid as 1m.



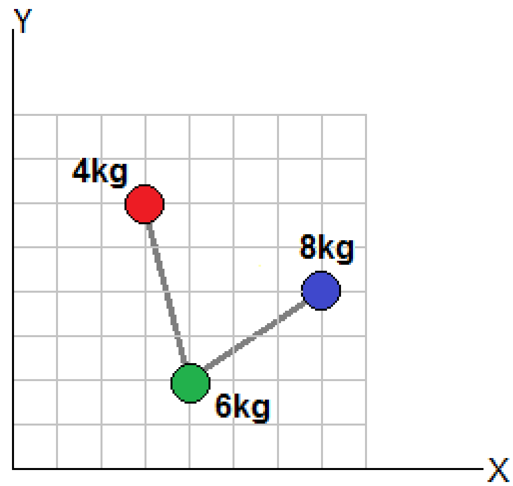
Solution:

Object	Mass	x	y	mx	my
red	2	3	6	6	12
green	5	4	2	20	10
blue	4	7	4	28	16
	11			54	38

$$X_{CM} = \frac{\sum m_i x_i}{\sum m_i} = \frac{54}{11} = \mathbf{4.9m}$$

$$Y_{CM} = \frac{\sum m_i y_i}{\sum m_i} = \frac{38}{11} = \mathbf{3.5m}$$

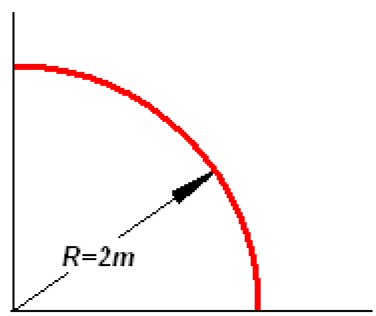
Problem 1a.- Find the center of mass X_{CM} and Y_{CM} of the following object made with three masses connected with thin rods.
Ignore the mass of the rods and take each square in the grid as 1m.



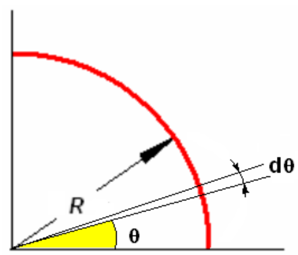
Solution: $X_{CM} = \dfrac{4\times3+6\times4+8\times7}{4+6+8} =$ **5.1 m**

$Y_{CM} = \dfrac{4\times6+6\times2+8\times4}{4+6+8} =$ **3.8 m**

Problem 2.- Find the center of mass of a wire with the shape shown in the figure.



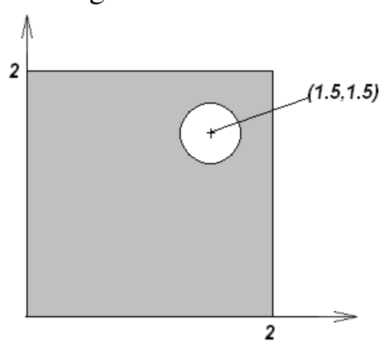
Solution: The mass of the infinitesimal arc is: $dm = \rho_L R d\theta$, where ρ_L is the linear density of the wire, R is the radius and $d\theta$ is the arc in radians.



Also, we notice that for that short piece of wire its center of mass is at $x = R\cos\theta$, and now we can integrate:

$$X_{CM} = \frac{\int x dm}{\int dm} = \frac{\int R\cos\theta \rho_L R d\theta}{\int \rho_L R d\theta} = \frac{R\int \cos\theta d\theta}{\int d\theta} = R\frac{2}{\pi} = \mathbf{1.27\ m}$$

Problem 3.- Find the center of mass of the plate shown in the figure, knowing that its thickness and density are uniform. It is a square of side L=2.00 with a circle cut out located at (1.50, 1.50) with a radius R=0.220 as shown in the figure. Give the answer with 3 significant figures.

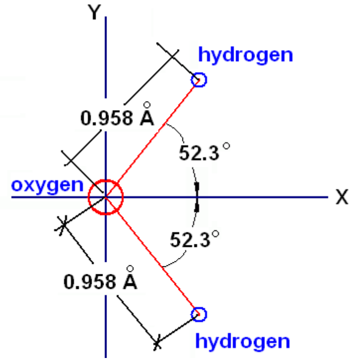


Solution: Consider the plate as made by two objects: A square of mass proportional to its area 2x2=4 and a circle of ***negative*** mass proportional to the area $-\pi R^2$, so the center of mass is:

$$X_{CM} = \frac{\sum x_i m_i}{\sum m_i} = \frac{1\times(2\times 2) + 1.5\times(-\pi R^2)}{(2\times 2) + (-\pi R^2)} = \frac{4 - 1.5\pi R^2}{4 - \pi R^2}$$

And with the values of the problem: $X_{CM} = \frac{4 - 1.5\pi(0.22)^2}{4 - \pi(0.22)^2} = \mathbf{0.980\ m}$

Problem 4.- Calculate the location of the center of mass of the water molecule if the mass of hydrogen is 1u and the mass of oxygen is 16u.



Solution: Notice that the oxygen atom is located at the origin, so its coordinates are (0,0), but the hydrogen atoms are located at:

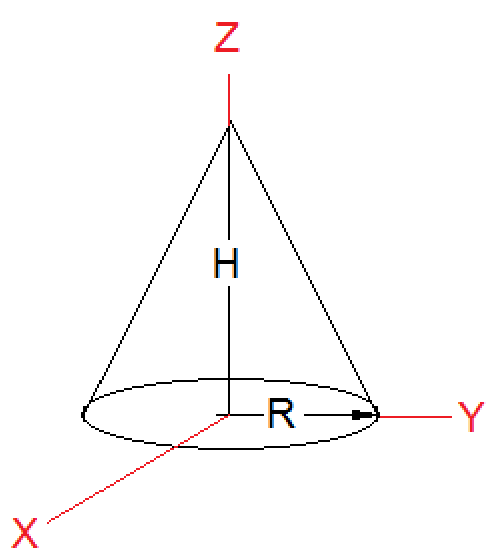
$$\vec{r}_1 = (0.958\,\text{Å}\cos(52.3^\circ), 0.958\,\text{Å}\sin(52.3^\circ))$$

$$\vec{r}_2 = (0.958\,\text{Å}\cos(52.3^\circ), -0.958\,\text{Å}\sin(52.3^\circ))$$

The vertical component of the center of mass is zero because of symmetry, and the horizontal component is:

$$X_{CM} = \frac{\sum m_i x_i}{\sum m_i} = \frac{m_O x_O + m_H x_1 + m_H x_2}{m_O + 2m_H} = \frac{16u(0) + 2u(0.958\,\text{Å}\cos 52.3^\circ)}{16u + 2u} = \mathbf{0.065\ \text{Å}}$$

Problem 5.- Calculate the location of the center of mass of the solid cone shown in the figure. Assume it has uniform density, radius at the base R=0.2m and height H=0.8m
Suggestion: You can integrate thin disks of radius “r” to get the center of mass. Notice that the radius would be r=R(1-z/H)

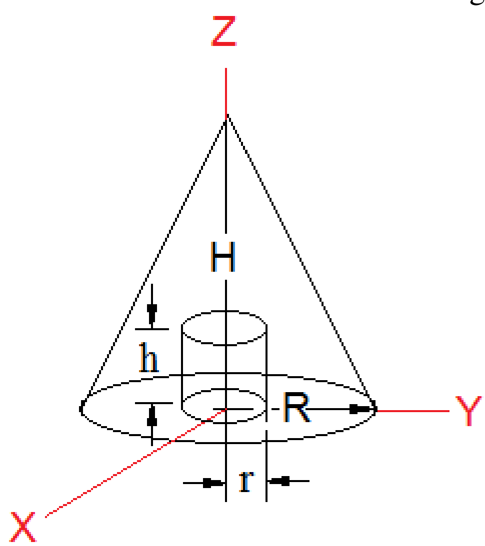


Solution: If we divide the cone in slices (disks) of radius $r = R(1 - z/H)$ and thickness dz then the mass of the disk will be: $dm = \rho\pi[R(1 - z/H)]^2 dz$, where ρ is the density of the cone. Now we can integrate to find the center of mass:

$$Z_{CM} = \frac{\int_0^H z dm}{\int_0^H dm} = \frac{\int_0^H z\rho\pi(R(1 - z/H))^2 dz}{\int_0^H \rho\pi(R(1 - z/H))^2 dz} = \frac{\int_0^H z(1 - z/H)^2 dz}{\int_0^H (1 - z/H)^2 dz} = \frac{H}{4}$$

Problem 5a: Calculate the location of the center of mass of the solid cone shown in the figure. Assume it has uniform density, radius at the base R=0.2m and height H=0.8m

There is a cylinder carved out at the base of radius r=0.1m and height h=0.2m



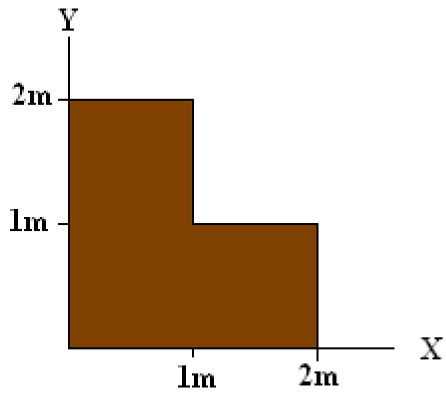
Solution: One possible way of solving the problem is by considering the carved out cylinder as a negative mass. The center of mass has to be on the z-axis due to symmetry, so we only need to find Z_{CM}.

Considering that the mass is proportional to the volume, we get:

$$Z_{CM} = \frac{\left(\frac{H}{4}\right)\left(\frac{1}{3}\pi R^2 H\right) + \left(\frac{h}{2}\right)\left(-\pi r^2 h\right)}{\left(\frac{1}{3}\pi R^2 H\right) + \left(-\pi r^2 h\right)} = \frac{R^2 H^2 - 6r^2 h^2}{4R^2 H - 12r^2 h} = \mathbf{0.22\ m}$$

We are using the result that we derived before that the center of mass of a solid cone is at ¼ of the height, measured from the base.

Problem 6.- Find the center of mass (x_{CM} and y_{CM}) of the following plate that has uniform thickness:



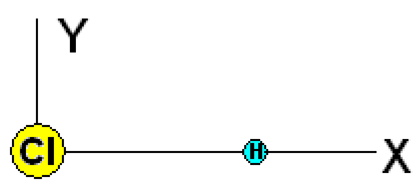
Solution: To find the center of mass (x_{CM} and y_{CM}) consider the plate made out of three squares with center of mass: (0.5,0.5) , (0.5,1.5) and (1.5, 0.5) and each of mass "M", then according to the equation for center of mass:

$$X_{CM} = \frac{\sum m_i x_i}{\sum m_i} = \frac{0.5M + 0.5M + 1.5M}{M + M + M} = \mathbf{0.83\ m}$$

$$Y_{CM} = \frac{\sum m_i y_i}{\sum m_i} = \frac{0.5M + 1.5M + 0.5M}{M + M + M} = \mathbf{0.83\ m}$$

Problem 7.- Calculate the location of the center of mass of the HCl molecule if the mass of hydrogen is 1u and the mass of chlorine is 35u and the distance between the atoms is 1.3Å. In your answer, specify where the center of mass is with respect to the chlorine nucleus.

Solution: Let us make the chlorine nucleus the origin of coordinates and the line connecting the two atoms the x-axis:



The vertical value of the center of mass is zero (because both atoms are located on the x-axis) and the x component is:

$$X_{CM} = \frac{\sum m_i x_i}{\sum m_i} = \frac{m_{Cl}x_{Cl} + m_H x_H}{m_{Cl} + m_H} = \frac{m_{Cl}(0) + m_H(1.3\,\mathring{A})}{m_{Cl} + m_H} = \frac{m_H(1.3\,\mathring{A})}{m_{Cl} + m_H}$$

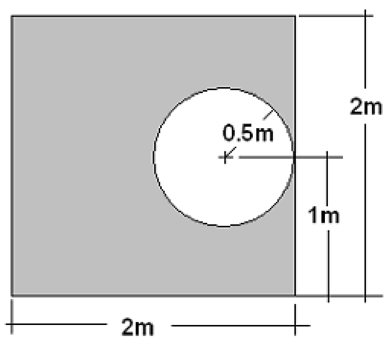
For $m_{Cl} = 35u$ we have $X_{CM} = \dfrac{1u(1.3\,\mathring{A})}{35u + 1u} = \mathbf{0.036\ \mathring{A}}$

Problem 8.- How far from the center of the Earth is the center of mass of the Earth-Moon system?
Consider the distance between the center of the Earth and the center of the Moon to be 384×10^3 km, the mass of the Earth=5.98×10^{24} kg and the mass of the Moon 7.35×10^{22} kg.

Solution: Taking the center of the Earth as the origin of coordinates:

$$X_{CM} = \frac{\sum m_i x_i}{\sum m_i} = \frac{(7.35\times10^{22})(384\times10^3 km)}{7.35\times10^{22} + 5.98\times10^{24}} = \mathbf{4,660\ km}$$

Problem 9.- A metallic plate with constant thickness and density has the shape of a square with a side of 2m. A round disk of radius 0.5m is removed from the square leaving the shape shown in the figure. Calculate where the center of mass is located.



Solution: If we set the origin of coordinates at the lower left corner, the object can be thought as composed of two objects:

i) A square of mass $m_1 = \rho t(2)^2$ where ρ is the density and t is the thickness.
With center of mass $x_1 = 1$

ii) A disk with negative mass $m_2 = -\rho t\pi(0.5)^2$ with center of mass $x_2 = 1.5$

This can be summarized in a table:

Object	Mass	X	miXi
Square	4	1	4
Circle	-0.25π	1.5	-0.375π

The center of mass is then:

$$x_{C.M.} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2} = \frac{4 - 0.375\pi}{4 - 0.25\pi} = \mathbf{0.88m}$$

In the vertical direction the center of mass is at $y_{C.M.} = \mathbf{1.0\ m}$

Problem 10.- Find how far from Pluto is the center of mass of the Pluto-Charon system. The mass of Pluto is 1.30×10^{22} kg, the mass of Charon is 1.51×10^{21} kg and the distance between them is 19,600 km.
Given that the radius of Pluto is 1,153 km, is the Center of Mass inside or outside Pluto?

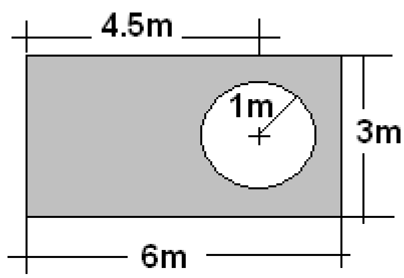
Solution: We can solve the problem preparing a table like the problems above.

object	*mass*	*x*	*mx*
Pluto	$1.30\times10^{22}kg$	0	0
Charon	$1.51\times10^{21}kg$	$19{,}600km$	$(1.51\times10^{21}kg)(19{,}600km)$
$\sum m =$	$1.451\times10^{22}kg$	$\sum mx =$	$(1.51\times10^{21}kg)(19{,}600km)$

$$X_{CM} = \frac{1.51\times10^{21}\times19{,}600}{1.451\times10^{22}} = \mathbf{2{,}040\ km}$$

This is outside Pluto!

Problem 11.- Find the center of mass of the following plate that has uniform thickness and density:



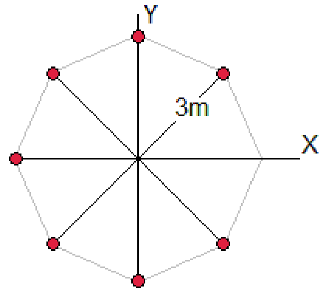
Solution: To solve this problem we can think of the figure as a rectangle plus a circle with negative mass. We do not know the masses, but the plate is uniform, so the masses are proportional to the areas. In fact, the mass can be calculated as area times surface density (ρ_s). With this in mind, the table in this case is the following:

	m_i	x_i
Rectangle	18	3
Circle	$-\pi$	4.5

Now we can calculate the center of mass of the figure:

$$X_{CM} = \frac{\sum m_i x_i}{\sum m_i} = \frac{18\times3-\pi\times4.5}{18-\pi} = \frac{54-4.5\pi}{18-\pi} = \mathbf{2.68\ m}$$

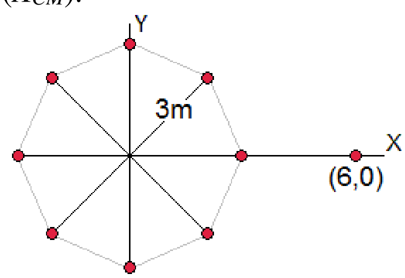
Problem 12.- Seven particles of mass 1kg each are located at the corners of an octagon as shown in the figure. Calculate the position of the center of mass (x_{CM}).



Solution: Consider the seven masses as a symmetric octagon minus one mass missing, so the center of mass is:

$$X_{CM} = \frac{\sum m_i x_i}{\sum m_i} = \frac{8\times 0 - 1\times 3}{8-1} = \frac{-3}{7} = \mathbf{-0.43\ m}$$

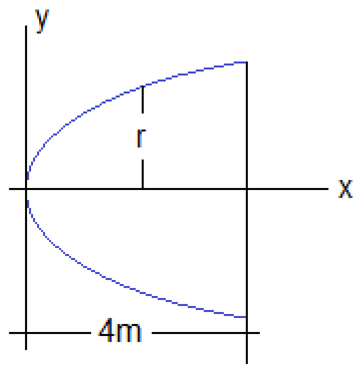
Problem 12a: A system of nine particles of mass 1kg each are located at the corners of an octagon, plus one particle at the (6,0) position as shown in the figure. Calculate the location of the center of mass of the system (X_{CM}).



Solution: Consider the nine masses as a symmetric octagon plus one mass at (6,0), so the center of mass is:

$$X_{CM} = \frac{\sum m_i x_i}{\sum m_i} = \frac{8\times 0 + 1\times 6}{8+1} = \frac{6}{9} = \mathbf{0.67\ m}$$

Problem 13.- A solid rocket nose has a shape that can be modeled as a cylinder with variable radius $r = 2\sqrt{x}$ as shown in the figure. Calculate the center of mass of the nose.
[Suggestion: divide the nose in infinitesimally thin disks of area πr^2]

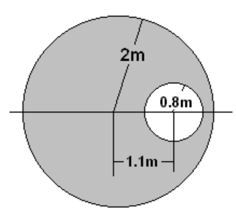


Solution: Using the suggestion of dividing the nose in slices, the mass of each slice will be $dm = \rho\pi r^2 dx$, where we multiply the density times the differential of volume, so:

$$X_{CM} = \frac{\int \rho\pi r^2 x dx}{\int \rho\pi r^2 dx}$$

But $r = 2\sqrt{x}$, so $X_{CM} = \frac{\int \rho\pi 4x x dx}{\int \rho\pi 4x dx} = \frac{\left.\frac{x^3}{3}\right|_0^4}{\left.\frac{x^2}{2}\right|_0^4} = \frac{8}{3} = \mathbf{2.7\ m}$

Problem 14.- Find the center of mass of the following plate that has uniform thickness and density. Take the center of the large circle to be the origin.



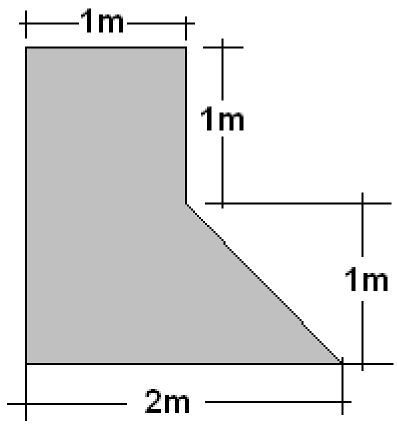
Solution: This can be summarized in a table:

Object	Mass	X	$m_i x_i$
Large circle	4πs	0	0
Hole	-0.64πs	1.1	-0.704π

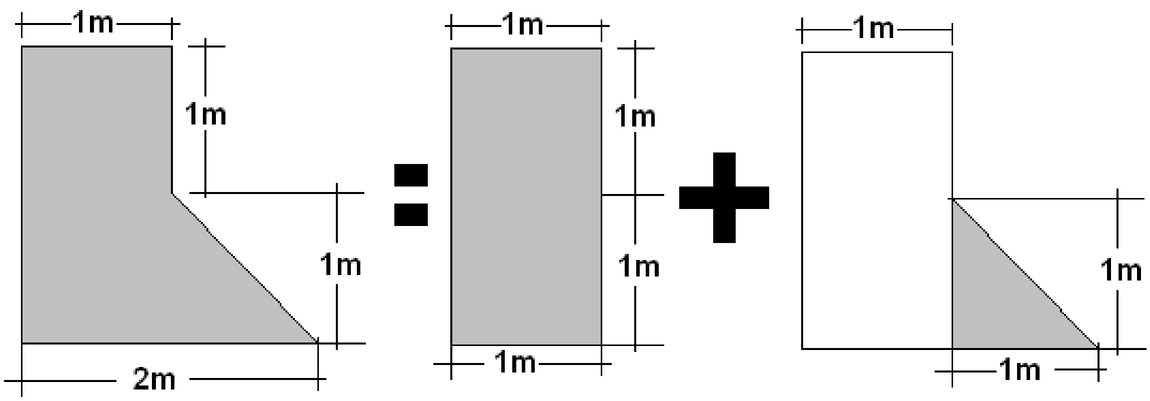
The center of mass is then:

$$x_{C.M.} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2} = \frac{-0.704\pi s}{4\pi s - 0.64\pi s} = \mathbf{-0.21\ m}$$

Problem 15.- Find the center of mass of the following plate that has uniform thickness and density:



Solution: To find the center of mass we divide the plate in two objects:



The rectangle has a mass proportional to its area: $m_1 = (2m)(1m)\rho t$ and has a center of mass at $x_1 = 0.5m$ and the triangle: $m_2 = \frac{(1m)(1m)}{2}\rho t$ and center of mass $x_2 = 1m + 0.33m$, so the center of mass is:

$$x_{C.M.} = \frac{\sum x_i m_i}{\sum m_i} = \frac{(2m)(1m)\rho t(0.5m) + \frac{(1m)(1m)}{2}\rho t(1.33m)}{(2m)(1m)\rho t + \frac{(1m)(1m)}{2}\rho t}$$

$$x_{C.M.} = \frac{1m + 0.5(1.33m)}{2 + 0.5} = \mathbf{0.667m}$$