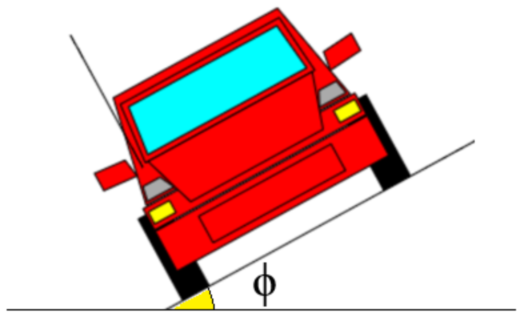
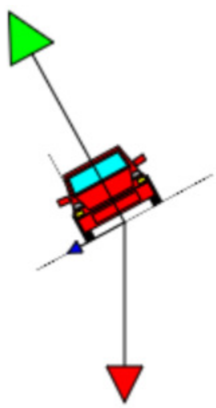
Physics I

Inclined Road

Problem 1.- A curve in a road has a radius R and an angle of inclination ϕ. What is the range of speeds for a car, so it does not slip when it takes the curve, if the coefficient of static friction between the tires and the pavement is μ_s?



Solution: Consider maximum safe speed first. In this case, friction prevents the car from slipping out of the curve, so the orientation of the force vectors, weight, normal and friction are as shown below in red, green, and blue.



With these conditions, the sum of the vertical forces must be zero, so

$$mg = -F_{friction} \sin\phi + F_{normal} \cos\phi$$

Since we are analyzing the extreme case, let us take $F_{friction} = \mu_s F_{normal}$, which replaced in the equation above gives us:

$$mg = -\mu_s F_{normal} \sin\phi + F_{normal} \cos\phi \qquad \text{Equation *}$$

Now we consider the horizontal direction. The sum of the forces must be equal to the centripetal force to make the car follow a circular trajectory of radius R, so we have:

$$\frac{mv^2}{R} = F_{friction} \cos\phi + F_{normal} \sin\phi$$

Again, we substitute $F_{friction} = \mu_s F_{normal}$

obtaining

$$\frac{mv^2}{R} = \mu_s F_{normal} \cos\phi + F_{normal} \sin\phi \qquad \text{Equation **}$$

We take equation ** and divide it, side by side, by equation *, obtaining:

$$\frac{\frac{mv^2}{R}}{mg} = \frac{\mu_s F_{normal} \cos\phi + F_{normal} \sin\phi}{-\mu_s F_{normal} \sin\phi + F_{normal} \cos\phi}$$

After simplifying and solving for the velocity we get:

$$v_{\max} = \sqrt{Rg \frac{\mu_s \cos\phi + \sin\phi}{-\mu_s \sin\phi + \cos\phi}}$$

This is the maximum speed without slipping out. Another version, dividing denominator and numerator by $\cos\phi$, is:

$$v_{\max} = \sqrt{Rg \frac{\mu_s + \tan\phi}{1 - \mu_s \tan\phi}}$$

It is interesting to note that the denominator might be negative. That would mean that the car will not slip at any arbitrary high speed, which happens if the tangent of the angle is greater than $1/\mu_s$.

Now, for the minimum speed, we notice that in this case the friction vector will point out of the curve. The equations are the same, except for the sign of the friction, giving us the minimum speed:

$$v_{\min} = \sqrt{Rg \frac{-\mu_s \cos\phi + \sin\phi}{\mu_s \sin\phi + \cos\phi}} \qquad \text{or} \qquad v_{\min} = \sqrt{Rg \frac{-\mu_s + \tan\phi}{1 + \mu_s \tan\phi}}$$

There is the possibility that the vehicle will not slip even if the speed is zero (if we just park it on the curve). For that to happen μ_s must be larger than the tangent of the angle.

We could also ask at what speed we do not need to rely on friction. That is, at what speed is the friction zero? At that point, replacing $F_{friction} = 0$ in the equations we get:

$$v_0 = \sqrt{Rg \tan\phi}$$

If we take the curve at higher speeds, friction will be towards the inside of the curve and if the speed is slower, friction will be towards the outside of the curve.

Problem 2.- A modern centaur (a biker riding a motorcycle) goes around a 35 m radius turn at 45 km/h. Find the angle of the banking, so friction won't be necessary to keep the creature from sliding.

Solution: The speed of the creature is:

$$v = 45\frac{km}{h}\left(\frac{1h}{3600s}\right)\left(\frac{1000m}{1km}\right) = 12.5m/s$$

The centripetal acceleration is: $a_R = \frac{v^2}{R} = \frac{(12.5m/s)^2}{35m} = 4.46m/s^2$

For friction to be unnecessary:

$$\tan\theta = \frac{a_R}{g} = \frac{4.46m/s^2}{9.8m/s^2} = 0.456 \rightarrow \theta = \mathbf{24.5^o}$$