Physics I

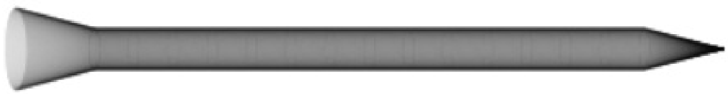
Rockets

$$F_{thrust} = v_{gases}\left|\frac{dm}{dt}\right|$$

Problem 1.- A solid rocket booster (SRB) has a thrust of 13.2×10^6 N. The total mass of the fuel is 500,000 kg and it is spent in 75 seconds.

(a) Calculate the speed of the gases expelled by the rocket.

(b) Calculate the acceleration produced at lift-off if the total mass is the mass of fuel plus 91,000 kg of "inert weight."



Solution:

According to the thrust equation: $F_{thrust} = v_{gases}\left|\frac{dm}{dt}\right|$, so the gases will be expelled with a speed given by the equation:

$$v_{gases} = \frac{F_{thrust}}{\left|\frac{dm}{dt}\right|},$$

Where the rate of fuel burning can be calculated using $\left|\frac{dm}{dt}\right| = \frac{m_{fuel}}{t}$, then:

$$v_{gases} = \frac{F_{thrust}t}{m_{fuel}}$$

Also, using Newton's second law we can find the acceleration produced by the thrust:

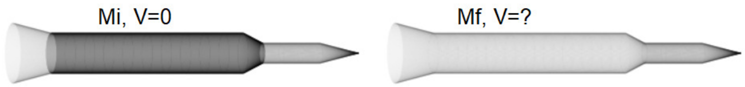
$a_{thrust} = \frac{F_{thrust}}{m_{fuel} + m_{inert}}$, at lift-off we need to subtract g from this quantity to get the true acceleration:

$$a = \frac{F_{thrust}}{m_{fuel} + m_{inert}} - 9.8$$

If $F_{thrust} = 13.2\times10^6$ N, $m_{fuel} = 500{,}000$ kg and t=75s then v_{gases} = **1,980 m/s**

And with an inert mass of 91,000 kg the acceleration is: $a =$ **12.5m/s²**

Problem 2.- A rocket has an initial mass Mi= 100,000 kg out of which 86,500 kg is fuel. The speed of the gases generated by burning the fuel is 2,700 m/s. Calculate the final speed of the rocket after burning all its fuel starting from zero velocity. Only consider the thrust of the gases, no other forces.



A) 675 m/s B) 1,350 m/s C) 2,700 m/s D) 5,400 m/s E) 8,100 m/s

Solution: The thrust is: $\mathrm{F} = -v_{gases}\dfrac{dm}{dt}$, but Newton's second law says:

$$\mathrm{F} = ma \rightarrow a = -\frac{v_{gases}\dfrac{dm}{dt}}{m}$$

$$\frac{\mathrm{dv}}{\mathrm{dt}} = -\frac{v_{gases}\dfrac{dm}{dt}}{m} \rightarrow \mathrm{dv} = -v_{gases}\frac{dm}{m}$$

Integrating on both sides of the equation we get:

$$\int_0^{\mathrm{v}} \mathrm{dv} = -v_{gases}\int_{100000}^{13,500}\frac{dm}{m}$$ and that gives: $v = v_{gases}\left[\ln(100,000) - \ln(13,500)\right] =$ **5,400 m/s**

Answer: D

Problem 3.- Calculate the speed of the gases of the Space Shuttle solid rocket booster that produce a thrust of $F_{thrust} = 12.5\times10^6 N$ and burns 499,000 kg of fuel in 110 seconds.

Solution: We know that thrust is equal to $F_{thrust} = \dfrac{dm}{dt}v_{relative}$

Given the values of the problem we can calculate the speed of the gases relative to the SRB as follows:

$$v_{relative} = \frac{F_{thrust}}{\left(\dfrac{dm}{dt}\right)} = \frac{12.5\times10^6 N}{499{,}00kg/110s} = $$ **2,755 m/s**

Problem 4.- A rocket uses solid fuel whose burned gases have a speed of 2700 m/s. At what rate do you need to burn the fuel to generate a thrust of 14,400 N?

Solution:

Since: $F_{thrust} = v_G\left|\dfrac{dm}{dt}\right| \rightarrow 14{,}400 = 2{,}700\left|\dfrac{dm}{dt}\right| \rightarrow \left|\dfrac{dm}{dt}\right| =$ **5.3 kg/s**

Problem 5.- A rocket has the following specs:

Initial mass = M_o =21,000 kg   Mass of fuel= M_{fuel} = 15,000 kg

Relative velocity of gasses: v_{gases} = 2800 m/s Rate of fuel burning: $\frac{dm}{dt}$ =-190kg/s

Find:

a) The thrust force
b) The initial acceleration

Solution Thrust force: $F = \frac{dm}{dt} v_{rel} = 2800 \times 190 =$ **532,000 N**

Initial acceleration: $a = \frac{F}{m} = \frac{532{,}000N}{21{,}000 + 15{,}000} =$ **14.7 m/s²**

Problem 6.- Given the following conditions, find the height reached by a rocket just after burning all the fuel assuming a vertical trajectory.

Initial mass = M_o = 21,000 kg   Mass of fuel= M_{fuel} = 15,000 kg

Relative velocity: v_{rel} =-2800 m/s   Rate of fuel burning: $\frac{dm}{dt}$ =-190kg/s

Solution*:* We can calculate the thrust by multiplying relative velocity times the rate of fuel burning:

$$F_{thrust} = v_{rel} \frac{dm}{dt} = 5.3 \times 10^5 N$$

If we want the net force, we need to subtract the weight:

$$F = F_{thrust} - Mg = F_{thrust} - \left(M_o + t \frac{dm}{dt} \right) g$$

At the instant of launch:

$F = F_{thrust} - M_o g = 3.2 \times 10^5 N$ This means an acceleration of $\frac{3.2 \times 10^5 N}{21{,}000 kg}$ =15.2 m/s^2

Just before burning all the fuel:

$$F = F_{thrust} - \left(M_o - M_{fuel} \right) g = 4.7 \times 10^5 N$$

This means a final acceleration of $\frac{4.7 \times 10^5 N}{6{,}000 kg}$ =78.3 m/s^2

This calculation raises the question: *Are we justified in using* g=9.8m/s^2 *as a constant?*

To calculate the velocity, we write:

$$F = M\frac{dv}{dt} = F_{thrust} - Mg \rightarrow \frac{dv}{dt} = \frac{F_{thrust} - Mg}{M} = \frac{F_{thrust}}{M_o + t\frac{dm}{dt}} - g$$

We multiply by *dt* and integrate:

$$\int_0^v dv = \int_0^t \frac{F_{thrust}dt}{M_o + t\frac{dm}{dt}} - \int_0^t gdt = F_{thrust}\left.\frac{\ln\left(M_o + t\frac{dm}{dt}\right)}{\frac{dm}{dt}}\right|_0^t - gt \rightarrow$$

$$v = \frac{F_{thrust}}{\frac{dm}{dt}}\ln\left(\frac{M_o + t\frac{dm}{dt}}{M_o}\right) - gt = v_{rel}\ln\left(1 + t\frac{1}{M_o}\frac{dm}{dt}\right) - gt$$

To calculate the velocity just after burning all the fuel we put t=$\frac{15,000kg}{190kg/s} = 78.9s$ in the equation above:

$$v = -2800\frac{m}{s}\ln\left(1 - \frac{78.9s}{110.5s}\right) - 9.8\frac{m}{s}(78.9s) = 2,730m/s$$

If we want the distance traveled, we can solve this last equation as follows:

$$\frac{dx}{dt} = v_{rel}\ln\left(1 + t\frac{1}{M_o}\frac{dm}{dt}\right) - gt \rightarrow x = v_{rel}\int_0^t \ln\left(1 + t\frac{1}{M_o}\frac{dm}{dt}\right)dt - \frac{1}{2}gt^2$$

Recall from calculus that:

$$\int_0^t \ln(1+ct)dt = \left(t + \frac{1}{c}\right)\ln(1+ct) - t$$, so: $$x = v_{rel}\left(t + \frac{M_o}{\frac{dm}{dt}}\right)\ln\left(1 + t\frac{1}{M_o}\frac{dm}{dt}\right) - v_{rel}t - \frac{1}{2}gt^2$$

With the values of the problem:

$$x = -2800\frac{m}{s}\left[(t - 110.5s)\ln\left(1 - \frac{t}{110.5s}\right) - t\right] - 4.9\frac{m}{s^2}t^2$$

To calculate the value of *x* just after burning all the fuel we put t=$\frac{15,000kg}{190kg/s} = 78.9s$ in the equation above:

$$x = -2800\frac{m}{s}\left[(78.9s - 110.5s)\ln\left(1 - \frac{78.9s}{110.5s}\right) - 78.9s\right] - 4.9\frac{m}{s^2}(78.9s)^2 = 79,700m$$

Now we can answer the question of whether we were justified in using g as a constant: At a height of 79,600m the acceleration due to gravity is:

$$g = G\frac{M_{Earth}}{(R_{Earth}+h)^2} = \left[G\frac{M_{Earth}}{R^2{}_{Earth}}\right]\frac{R^2{}_{Earth}}{(R_{Earth}+h)^2} = g\,\frac{R^2{}_{Earth}}{(R_{Earth}+h)^2} = g\,\frac{1}{\left(1+\frac{h}{R_{Earth}}\right)^2}$$

So, g is: $\frac{9.8m/s^2}{\left(1+\frac{79{,}700m}{6'380{,}000m}\right)^2} = 9.56m/s^2$, a 2.5% correction.