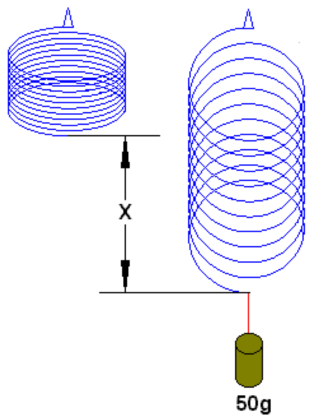
Physics I

Simple Harmonic Oscillator

$F = -kx$ Hooke's law

$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$ and $T = 2\pi\sqrt{\frac{m}{k}}$ for a simple harmonic oscillator

Problem 1.- You hang a 50gram mass from a spring and it stretches x = 25cm, then you pull the mass 5 cm from equilibrium and release it. Calculate the period of the oscillation.
Ignore the mass of the spring.



Solution: The spring constant is $k = \frac{F}{x} = \frac{0.05 \times 9.8}{0.25} = 1.96\text{ N/m}$

And the period is: $T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{0.05}{1.96}} = \mathbf{1.00\ s}$

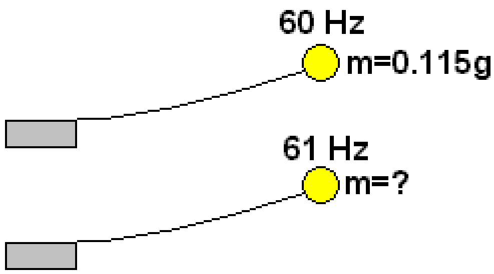
Problem 1a.- A fisherman's scale stretches 2.5cm when a 2.5-kg fish hangs from it.
(a) Calculate the spring constant of the scale in N/m.
(b) What will be the frequency of oscillation if you pull the fish down and release it?

Solution: (a) The spring constant is given by: $k = \frac{mg}{x} = \frac{2.5kg(9.8m/s^2)}{0.025m} = \mathbf{980\ N/m}$

(b) The frequency:

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}} = \frac{1}{2\pi}\sqrt{\frac{980\text{N/m}}{2.5\text{kg}}} = \mathbf{3.15Hz}$$

Problem 2.- A frequency indicator has a spring leaf of negligible mass itself with a mass of 0.115 grams attached at its end. Its natural oscillation frequency in this way is 60.0 Hz.
Calculate the mass that you would need for 61.0 Hz.
Hint: Notice that "k" is the same in both cases.



Solution: Since $f = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$ then $k = 4\pi^2 f^2 m$ and it is the same in both cases, so:

$$4\pi^2 f_1^2 m_1 = 4\pi^2 f_2^2 m_2 \rightarrow f_1^2 m_1 = f_2^2 m_2 \rightarrow m_2 = \frac{f_1^2 m_1}{f_2^2} = \frac{60^2 0.115}{61^2} = \mathbf{0.111\ g}$$

Problem 3.- True (T) or false (F) about the simple harmonic oscillator:

(**F**) If you double the amplitude, the period doubles.
(**F**) If you double the mass, the period doubles.
(**T**) If you quadruple the mass, the period doubles.
(**T**) If you increase the stiffness of the spring (increase k), the period decreases.
(**T**) The acceleration in the middle is zero.

Problem 4.- At what displacement x will a simple harmonic oscillator reach 80% of its maximum velocity? Give your answer as a fraction of the amplitude A.

Solution: If the velocity is 80% of the maximum:

$\frac{1}{2}kA^2 = \frac{1}{2}kx^2 + \frac{1}{2}m(0.8v_{max})^2$, but $\frac{1}{2}kA^2 = \frac{1}{2}m_{max}^2$, so:

$$\frac{1}{2}kA^2 = \frac{1}{2}kx^2 + \frac{1}{2}0.64kA^2 \rightarrow A^2 = x^2 + 0.64A^2 \rightarrow \boldsymbol{x = 0.6\,A}$$

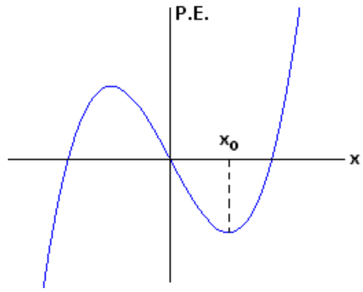
Problem 4a.- What is the velocity of a simple harmonic oscillator when its position is one third the amplitude ($x=A/3$)? Give your answer as a fraction of the maximum velocity.

Solution: If the position is $1/3A$:

$\frac{1}{2}kA^2 = \frac{1}{2}k(A/3)^2 + \frac{1}{2}mv^2 \rightarrow \frac{8kA^2}{9} = mv^2$, but $\frac{1}{2}kA^2 = \frac{1}{2}mv_{max}^2$, so:

$$\frac{8}{9}kA^2 = \frac{8}{9}mv_{max}^2 = mv^2 \rightarrow \quad v = \frac{2\sqrt{2}}{3}v_{max}$$

Problem 5.- A particle of mass **m=2kg** is trapped in the potential: $\text{P.E.} = x^3 - 4x$
Find the value of x_o, where the potential has a local minimum and find the period of small oscillations of the particle around that point.
[Remember that $\left.\frac{d^2\text{P.E.}}{dx^2}\right|_{x_o}$ plays the role of "k" in the simple harmonic oscillator]



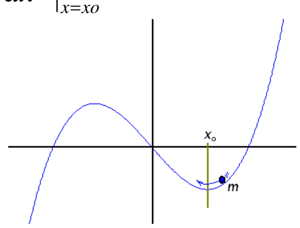
Solution: First let's find the value of x_o: $\frac{d\text{P.E.}}{dx} = 3x^2 - 4 = 0 \rightarrow x_o = \frac{2}{\sqrt{3}}$

Then we find $k = \left.\frac{d^2P.E.}{dx^2}\right|_{x_o} = 6x = 6\frac{2}{\sqrt{3}} = 4\sqrt{3}$

Finally, we find the period: $T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{2}{4\sqrt{3}}} = \frac{2\pi}{\sqrt{2\sqrt{3}}} =$ **3.4 s**

Problem 5a.- The potential:
$V = x^3 - 3x$
Where V is in joules and x in meters, has a minimum at $x_o = 1$
A particle of mass *m=1kg* is trapped by the potential. Find the angular frequency of small oscillations around x_o

Recall that $f = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$ *and* $k = \left.\frac{d^2V}{dx^2}\right|_{x=xo}$



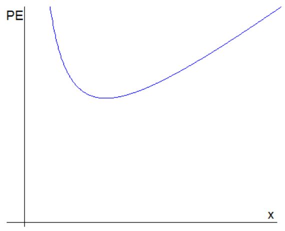
Solution: The first derivative of the potential is $\frac{dV}{dx} = 3x^2 - 3$, which indeed is zero at $x_o = 1$, now we find the second derivative $\frac{d^2V}{dx^2} = 6x$, which evaluated at $x_o = 1$ is 6.

So: $f = \frac{1}{2\pi}\sqrt{\frac{6}{1}} =$ **0.39 Hz**

Problem 5b.- A particle of mass **m=0.48kg** is trapped in the potential: $P.E. = 4x + \frac{9}{x}$

Find the value of x_o, where the potential has a local minimum and find the period of small oscillations of the particle around that point.

[Remember that $\left.\frac{d^2P.E.}{dx^2}\right|_{x_o}$ plays the role of "k" in the simple harmonic oscillator]



Solution: First let's find the value of x_o: $\frac{dP.E.}{dx} = 4 - \frac{9}{x^2} = 0 \rightarrow x_o = \sqrt{\frac{9}{4}} = 1.5$

Then we find $k = \left.\frac{d^2P.E.}{dx^2}\right|_{x_o} = \frac{18}{x^3} = \frac{18}{1.5^3} = 5.33$

Finally, we find the period: $T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{0.48}{5.33}} = 0.6\pi =$ **1.88 s**

Problem 6.- The vibrational frequency of H_2 is 1.32×10^{14} Hz. So, how much is the vibrational frequency of D_2 (deuterium) that has the same "spring constant" but twice the mass?

Solution: The frequency for hydrogen is: $1.32\times10^{14} = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$

And for deuterium it is: $f = \frac{1}{2\pi}\sqrt{\frac{k}{2m}}$

Dividing the second equation by the first we get:

$\frac{f}{1.32\times10^{14}} = \sqrt{\frac{1}{2}} \rightarrow f = \frac{1.32\times10^{14}}{\sqrt{2}} =$ **9.33×10^{13} Hz**

Problem 6a.- The vibrational frequency of H_2 is 1.32×10^{14} Hz. So, how much is the vibrational frequency of T_2 (tritium) that has the same "spring constant" but three times the mass?

Solution: Similar to the previous problem

$f = \frac{1.32\times10^{14}}{\sqrt{3}} =$ **7.62×10^{13} Hz**

Problem 7.- A supermarket's scale in the produce section stretches 5.5cm when a 2.5-kg watermelon is added to the plate. If the mass of the plate is 0.5kg:
(a) Calculate the spring constant of the scale in N/m.
(b) What will be the frequency of oscillation if you push the watermelon down and release it?

Solution:
(a) To calculate the spring constant of the scale, notice that the external force is the weight mg = 2.5kg (9.8m/s^2) and it generates a displacement of 5.5cm = 0.055m, so:

$$F = -kx \rightarrow k = -\frac{F}{x} = \frac{2.5kg(9.8m/s^2)}{0.055m} = \mathbf{445\ N/m}$$

(b) To find the frequency of oscillation we use: $f = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$, but be careful to use the total oscillating mass: watermelon plus plate:

$$f = \frac{1}{2\pi}\sqrt{\frac{445\text{N/m}}{3\text{kg}}} = \mathbf{1.9\ Hz}$$

Problem 7a.- The springs of an 800-kg car compress 5 mm when a 75-kg driver gets into the driver's seat. Calculate the frequency of vibration after hitting a bump (assume there are no shock absorbers).

Solution: We can first find the spring constant by noticing that the compression was 5 mm for a weight of a 75-kg person:

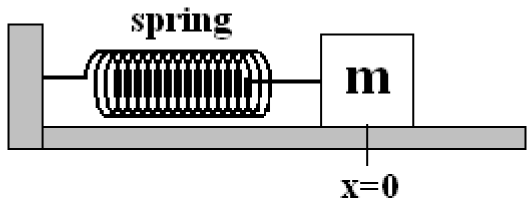
$$F = -kx \rightarrow k = -\frac{F}{x} = \frac{75\text{kg}(9.8\text{m/s}^2)}{0.005\text{m}} = 147{,}000\ \text{N/m}$$

Notice that we calculated the weight by multiplying the mass of the person times the acceleration of gravity. Also notice that we converted x=5 mm to x=0.005 m.

With this value of "k" we can find the frequency, just be careful to use the total mass (car plus person) in the formula:

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}} = \frac{1}{2(3.1416)}\sqrt{\frac{147000\text{N/m}}{875\text{kg}}} = \mathbf{2.1\ Hz}$$

Problem 8.- The mass shown in the figure, which was in equilibrium is pulled to the right and released. There is negligible friction between the block and the surface, so the mass oscillates back and forth.



a) Where is the acceleration of the block maximum?

b) Where is the velocity of the block maximum?

c) Is the acceleration of the block ever zero? if so, where?

d) Is the velocity of the block ever zero? if so, where?

Solution:

a) The acceleration of the block is a maximum at the end of the motion.

b) The velocity is a maximum in the middle of the oscillation.

c) The acceleration is zero when the mass is at the point x=0, because the force is zero there (remember that F=kx)

d) The velocity of the block is zero at the two turning points.

Problem 9.- A bungee jumper of mass 60 kg jumps from a high bridge and oscillates up and down with a frequency of 0.5 Hz. Calculate the frequency of oscillation for another jumper of mass 80 kg that uses the same bungee cord.

Solution: Translating the data of the problem into equations we can write:

$$f_1 = 0.5\text{Hz} = \frac{1}{2\pi}\sqrt{\frac{k}{60\text{kg}}}$$

$$f_2 = \frac{1}{2\pi}\sqrt{\frac{k}{80\text{kg}}}$$

We could find the spring constant of the bungee rope using the first equation and then use it in the second equation to find the frequency, instead it is easier to just divide one equation by the other:

$$\frac{f_2}{f_1} = \frac{f_2}{0.5\text{Hz}} = \frac{\frac{1}{2\pi}\sqrt{\frac{k}{80\text{kg}}}}{\frac{1}{2\pi}\sqrt{\frac{k}{60\text{kg}}}}$$

Almost all terms cancel and what is left allows us to find the frequency of the second jumper:

$$\frac{f_2}{0.5\text{Hz}} = \sqrt{\frac{60\text{kg}}{80\text{kg}}} \rightarrow f_2 = 0.5\text{Hz}\sqrt{\frac{60\text{kg}}{80\text{kg}}} = \mathbf{0.43\ Hz}$$

Problem 9a.- A fly of mass 0.025g is caught in a spider's web. The web vibrates with a frequency of 4.5Hz. What would be the frequency if the mass of the fly were 0.049g? Assume the web behaves like an ideal spring, making this a case of simple harmonic motion.

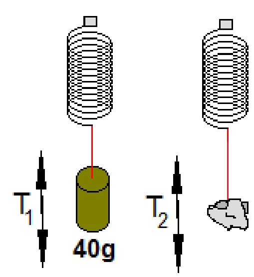
Solution: Given the equation of the Simple Harmonic Oscillator: $f = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$, we can write:

$$4.5\text{Hz} = \frac{1}{2\pi}\sqrt{\frac{k}{0.025\text{g}}} \quad \textit{and} \quad f_2 = \frac{1}{2\pi}\sqrt{\frac{k}{0.049\text{g}}}$$

Dividing the second equation by the first we get:

$$\frac{f_2}{4.5\text{Hz}} = \frac{\frac{1}{2\pi}\sqrt{\frac{k}{0.049\text{g}}}}{\frac{1}{2\pi}\sqrt{\frac{k}{0.025\text{g}}}} = \frac{5}{7} \rightarrow f_2 = \mathbf{3.21\ Hz}$$

Problem 10.- In the lab you hang 40 grams from a spring and it oscillates with a period T_1=1.2 seconds, then you replace the mass with an unknown rock and the period is now T_2=0.6 seconds. What is the mass of the rock?



Solution: We write two equations for the periods

$$T_1 = 2\pi\sqrt{\frac{m_1}{k}}$$

$$T_2 = 2\pi\sqrt{\frac{m_2}{k}}$$

Then we divide one equation by the other $\frac{T_2}{T_1} = \frac{2\pi\sqrt{\frac{m_2}{k}}}{2\pi\sqrt{\frac{m_1}{k}}} = \sqrt{\frac{m_2}{m_1}}$, now we can solve for the mass of the rock $m_2 = m_1\left(\frac{T_2}{T_1}\right)^2 = 40\times\left(\frac{0.6}{1.2}\right)^2 =$ **10g**

Problem 11.- An industrial separator shakes coffee beans by vibrating a table vertically following the equation $y = 0.15\sin\omega t$
Calculate the value of ω, so the beans lose contact with the table.

Solution: Since $y = 0.15\sin\omega t$ then $v_y = \frac{dy}{dt} = 0.15\omega\cos\omega t$ and $a_y = \frac{d^2y}{dt^2} = -0.15\omega^2\sin\omega t$, whose maximum value is $0.15\omega^2$, this has to match g=9.8m/s^2 if the beans are going to lose contact, so:

$0.15\omega^2 = 9.8 \rightarrow \omega = \sqrt{\frac{9.8}{0.15}} =$ **8.1 rad/s**

Problem 12.- The position of a simple harmonic oscillator is given by the equation:

$x = 5.5\,\mathrm{Sin}(4\pi t)$

Where x is in meters and t is in seconds.
(a) Calculate the period and frequency
(b) What will be the velocity and acceleration at t = 2.0s?

Solution:

(a) We notice that the argument of the sine function is:

$4\pi t = 2\pi f t \rightarrow f = 2\mathrm{Hz} \rightarrow T = 0.5\mathrm{s}$

(b) To get the velocity we can take the derivative once:

$$v = \frac{dx}{dt} = 5.5(4\pi)\mathrm{Cos}(4\pi t) \rightarrow v(2s) = 5.5(4(3.1416))\mathrm{Cos}(8\pi) = 69.1m/s$$

To get the acceleration we derive a second time:

$$a = \frac{d^2x}{dt^2} = -5.5(2\pi)^2\sin(4\pi t) \rightarrow a(2s) = 5.5(2(3.1416))\sin(8\pi) = 0m/s^2$$

Problem 13.- The position of a simple harmonic oscillator is given by the equation:

$$x = 8.5\sin(4\pi t)$$

Where x is in meters and t is in seconds.
Calculate:
(a) Period
(b) Frequency
(c) Maximum speed
(d) Maximum acceleration

Solution:

(a) We can identify the argument of the sine function

$$4\pi t = \frac{2\pi}{T}t$$

So the period is T = **0.5s**

(b) The frequency is the inverse of the period f = **2.0Hz**

(c) The speed is

$$v = \frac{dx}{dt} = 8.5\times 4\pi\cos(4\pi t)$$

The maximum value is $v_{max} = 8.5\times 4\pi =$ **107m/s**

(d) The acceleration is

$$a = \frac{dv}{dt} = -8.5\times 4\pi\times 4\pi\sin(4\pi t)$$

The maximum value is $a_{max} = 8.5\times 4\pi\times 4\pi =$ **1,342m/s^2**