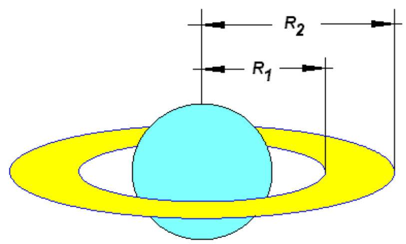
Physics I

Kepler's Laws

Kepler's third law:

$$\frac{R_1^{\,3}}{T_1^{\,2}} = \frac{R_2^{\,3}}{T_2^{\,2}}$$

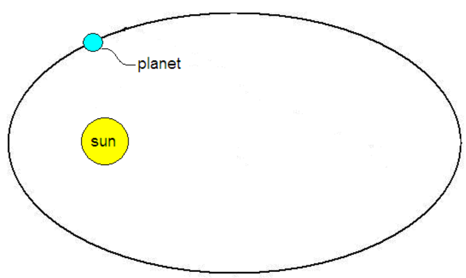
Problem 1.- A planet has a ring of particles describing circular orbits. The outer part of the ring has a radius R_2=27,500 km and a period of 12.5 days. Calculate the period of the particles in the inner part of the ring where the radius is R_1=17,600 km



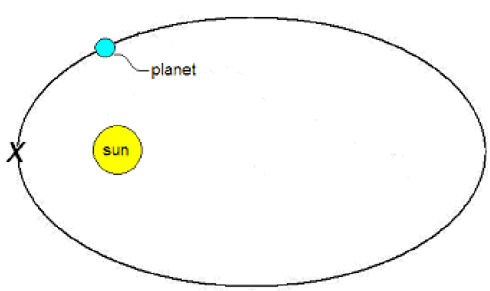
Solution: Using Kepler's third law:

$$\frac{R_1^{\,3}}{T_1^{\,2}} = \frac{R_2^{\,3}}{T_2^{\,2}} \rightarrow \frac{17{,}600^3}{T_1^{\,2}} = \frac{27{,}500^3}{12.5^2} \rightarrow T_1 = 12.5\sqrt{\frac{17{,}600^3}{27{,}500^3}} = \mathbf{6.4\ days}$$

Problem 2.- Indicate at what point in the elliptical orbit the planet has its maximum kinetic energy. Give a rationale for your answer.



Solution: The maximum velocity happens at closest approach, so that is the point with maximum kinetic energy too.



Problem 3.- A satellite of mass *m* orbits a planet of mass *M* in a circular orbit with radius *R*. The time required for one revolution is:

(A) Independent of *M*
(B) Proportional to *m*
(C) Linear in *R*
(D) Proportional to $R^{3/2}$
(E) Proportional to R^2

Solution: Since the centripetal force has to be equal to the gravitational force, we have:

$$F = G\frac{Mm}{R^2} = mR\omega^2 \rightarrow G\frac{M}{R^2} = R\omega^2 \rightarrow G\frac{M}{R^2} = R\frac{4\pi^2}{T^2}, \text{ which gives us: } T^2 = \frac{4\pi^2}{GM}R^3$$

(A) This is incorrect because the period does depend on the mass of the planet.

(B) The period does depend on the satellite mass, but it is through the "reduced" mass of the system, which is normally a very small effect that was neglected in the equation above. In any case, the period is not proportional to that mass, so this statement is incorrect.

(C) Incorrect because it is not a linear relation between T and R.

(D) Correct answer. The period is proportional to $R^{3/2}$

(E) Incorrect, the relation between T and R is not quadratic.

Problem 4.- Callisto is one of the moons of Jupiter that was discovered by Galileo. It has a period of 16.7 days and a mean distance from Jupiter of 1.883×10^6 km. Calculate the period of Io, which is also a moon of Jupiter, knowing that its mean distance to Jupiter is 0.422×10^6 km.

Solution: According to Kepler's third law: $\frac{T^2}{R^3} = \text{constant}$ for this Jovian system, so with the values of the problem:

$$\frac{T_{Callisto}^{\ 2}}{R_{Callisto}^{\ 3}} = \frac{T_{Io}^{\ 2}}{R_{Io}^{\ 3}} \rightarrow T_{Io} = T_{Callisto}\sqrt{\left(\frac{R_{Io}}{R_{Callisto}}\right)^3} = 16.7\sqrt{\left(\frac{0.422\times10^6}{1.883\times10^6}\right)^3} = \mathbf{1.77\ days}$$

This is equal to the measured value of 1.77 days.

Problem 5.- An artificial satellite in orbit around our planet needs 12 hours to complete one revolution. Calculate the radius of the circular orbit knowing that the period of a satellite in low earth orbit (LEO) is approximately 90 minutes.

LEO is an orbit so close to the Earth that you can say that the radius of the orbit is equal to the radius of the earth R_E.

Solution:

$$\frac{T_1^{\ 2}}{R_1^{\ 3}} = \frac{T_2^{\ 2}}{R_2^{\ 3}} \rightarrow \frac{720^2}{R_{satellite}^{\ \ 3}} = \frac{90^2}{R_{Earth}^{\ \ 3}} \rightarrow R_{satellite} = R_{Earth}\sqrt[3]{\frac{720^2}{90^2}} = 4R_{Earth}$$

Problem 6.- Calculate how long the Martian year is, knowing that its average distance to the Sun is 228×10^6 km.

Earth's year is 365.25 days and the distance from the Earth to the Sun is 150×10^6 km.

Solution: Using Kepler's 3rd law:

$$\frac{T_{MARS}^{\ \ 2}}{228^3} = \frac{365.25^2}{150^3} \rightarrow T_{MARS} = 365.25\sqrt{\frac{228^3}{150^3}} = \mathbf{684\ days}$$