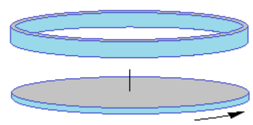
Physics I

Angular Momentum

Problem 1.- A uniform disk is rotating at a steady angular velocity of 0.87 rad/s (ignore friction in the axle). You then drop a hoop of the same mass and radius on top of the disk. What is the angular velocity now?

$I_{HOOP} = mR^2$

$I_{DISK} = \frac{1}{2}mR^2$



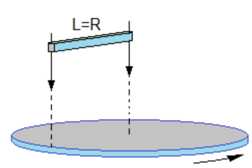
Solution: Newton's third law of motion applied to an isolated rotating system tells us that the total angular momentum should remain the same.

Before dropping the hoop, the angular momentum was $\left(\frac{1}{2}mR^2\right)\omega_1$ and after the drop, when the disk *and* hoop rotate together, the angular momentum is $\left(\frac{1}{2}mR^2 + mR^2\right)\omega_2$. These must be the same, so:

$$\left(\frac{1}{2}mR^2\right)\omega_1 = \left(\frac{1}{2}mR^2 + mR^2\right)\omega_2 \rightarrow \omega_2 = \frac{\omega_1}{3} = \mathbf{0.29\ rad/s}$$

Problem 1a.- A uniform disk of mass 1.0kg and radius 1.0m is rotating freely (ignore friction in the axle) at a steady angular velocity of 1.44rad/s. You then drop on top of the disk a thin bar of mass 1.0kg and length 1.0m, as shown in the figure, and they rotate together. What is the angular velocity now?

$I_{bar-rotated-at-one-end} = \frac{1}{3}mL^2$ $\qquad$ $I_{disk} = \frac{1}{2}mR^2$

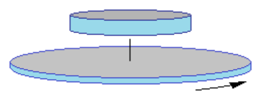


Solution: Like in the previous problem, the total angular momentum should remain the same. Before dropping the bar, the angular momentum was $\left(\frac{1}{2}mR^2\right)\omega_1$ and after the drop, when the disk *and* bar rotate together, the angular momentum is $\left(\frac{1}{2}mR^2+\frac{1}{3}mR^2\right)\omega_2$. These must be the same, so: $\left(\frac{1}{2}mR^2\right)\omega_1=\left(\frac{1}{2}mR^2+\frac{1}{3}mR^2\right)\omega_2 \to \omega_2=\frac{3\omega_1}{5}=$ **0.86 rad/s**

Problem 1b.- A uniform disk of radius 1m and mass 1kg is rotating at a steady angular velocity of 1.25 rad/s. You then drop another disk of mass 1kg too, but radius 0.5m on top of the rotating disk.

What is the angular velocity now?

$$I_{DISK}=\frac{1}{2}mR^2$$



Solution: This is one more application like the previous two problems where the total angular momentum should remain the same.

Before dropping the smaller disk, the angular momentum was

$$\left(\frac{1}{2}mR^2\right)\omega_1=\left(\frac{1}{2}(1)(1)^2\right)1.25=0.625\text{kgm}^2\,\frac{\text{rad}}{\text{s}}$$

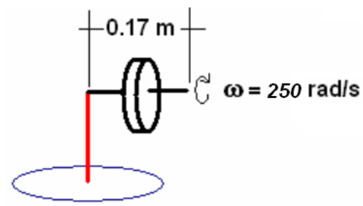
And after the drop, when the two disks rotate together the angular momentum is

$$\left(\frac{1}{2}(1)(1)^2+\frac{1}{2}(1)(0.5)^2\right)\omega_2=0.625\text{kgm}^2\omega_2\,,$$

These must be the same, so:

$$0.625\text{kgm}^2\,\frac{\text{rad}}{\text{s}}=0.625\text{kgm}^2\omega_2 \to \omega_2=\textbf{1.0 rad/s}$$

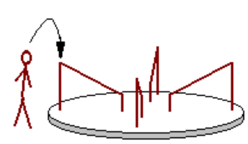
Problem 2.- Consider a gyroscope consisting of a disk of mass 0.25kg and radius 0.055m mounted at the center of an axle 0.17m long. The gyroscope spins at 250 radians/s. Calculate how long it takes for the gyroscope to precess once around.



Solution: The gyroscope has an angular momentum $L = I\omega = \frac{1}{2}MR^2\omega$ and the torque due to its weight is: $\tau = \frac{dL}{dt} = Mg\frac{x}{2}$, where x is the length of the axle. To complete one turn we need for the angular momentum vector to rotate once full circle whose circumference is $2\pi L$, so: $\frac{dL}{dt}T = 2\pi L$.

$$\rightarrow T = \frac{2\pi L}{\left(\frac{dL}{dt}\right)} = \frac{2\pi\left(\frac{1}{2}MR^2\omega\right)}{Mg\frac{x}{2}} = \frac{\pi R^2\omega}{gr} = \frac{(3.1416)(0.055m)^2(250rad/s)}{(9.8m/s^2)(0.085m)} = \mathbf{2.85\ s}$$

Problem 3.- A 3.0m-radius merry-go-round is rotating freely at 1.2 rad/s. Its moment of inertia is 1500kgm^2. A person of mass 75kg suddenly steps on the edge of the merry-go-round. What is the angular velocity now?



Solution: Since there is no external torque, the angular momentum is conserved:

$I_1\omega_1 = I_2\omega_2$

The moment of inertia before the person steps on the merry-go-round is only 1,500kgm^2, but after the person steps on the edge we need to add the moment of inertia of the person:

$$I_2 = 1500kgm^2 + 75kg(3m)^2 = 2175kgm^2$$

Here we made the approximation that all the mass of the person is 3m from the center of the merry-go-round.

The new angular velocity is: $\omega_2 = \frac{I_1\omega_1}{I_2} = \frac{1{,}500}{2{,}175}1.2\frac{\text{rad}}{\text{s}} =$ **0.83 rad/s**

Problem 3a.- A merry-go-round rotates at 30 rpm with a girl sitting on the edge. What would happen if the child walked to the center of the disk? Neglect friction and assume there is no other external torque.

Solution: Since there is no external torque, the product *I* times angular velocity will stay the same. When the child moves to the center, *I* will diminish and so the angular velocity will increase.

Problem 4.- A springboard diver starts her motion rotating at a rate of four turns per second with her arms and body contracted, and then she stretches, doubling her moment of inertia. What is the new rotational speed?

Solution: When in free fall, the diver conserves angular momentum about her center of mass (since her weight produces zero torque) so:

$$L = I_1\omega_1 = I_2\omega_2$$

If the moment of inertia doubles, the angular velocity will need to drop to one half to compensate.
If the initial angular velocity is 4 turns per second before, it will be **2 turns per second** after.

Problem 5.- Two particles with mases m_1 = 2 kg and m_2 = 3 kg, move according to the trajectories

$r_1 = (t^2, t^3, 0)$

$r_2 = (5+2t^2, -t^3, 0)$

Here, t is in seconds and the distances are in meters. Calculate

(a) The external resultant force acting on the system.

(b) The total external torque acting on the system.

(c) The angular momentum of the system.

Solution: The force is

$$\vec{F} = \sum m\vec{a} = 2(2,6t,0) + 3(4,-6t,0) = (16,-6t,0)$$

The torque

$$\vec{\tau}=\sum \vec{r}\times m\vec{a}=2\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ t^2 & t^3 & 0\\ 2 & 6t & 0\end{vmatrix}+3\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 5+2t^2 & -t^3 & 0\\ 4 & -6t & 0\end{vmatrix}=(-16t^3-90t)\hat{k}$$

And the angular momentum

$$\vec{L}=\sum \vec{r}\times m\vec{v}=2\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ t^2 & t^3 & 0\\ 2t & 3t^2 & 0\end{vmatrix}+3\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 5+2t^2 & -t^3 & 0\\ 4t & -3t^2 & 0\end{vmatrix}=(-4t^4-45t^2)\hat{k}$$

Problem 6.- If all the ice in the poles would melt and distribute in the oceans, would the days be longer, shorter or the same?

Solution: The moment of inertia of the Earth would increase because the water from the melted poles would be further away from the axis of the planet. To conserve angular momentum, the angular velocity would diminish. The period (the days) would be longer.