Physics I

Moment of Inertia

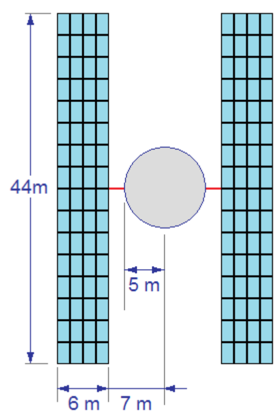
Moment of inertia for a distributed mass $I = \int r^2 dm$

Moment of inertia for discrete point masses $I = \sum m_i r_i^2$

Moment of inertia of a disk about its center $I = \frac{1}{2} mR^2$

Parallel axes theorem $I = I_{CM} + m\ell^2$

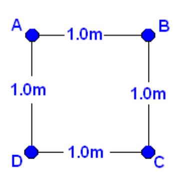
Problem 1.- A mini space station (MSS) can be modeled as a hollow sphere with radius R = 5m and mass 2,400 kg and two rectangular solar panels of mass 1,200 kg each, with the dimensions shown in the figure. Calculate the moment of inertia of the MSS with respect to an axis that passes through the center of the two solar panels.



Solution: The moment of inertia of a hollow sphere (or shell) $I = \frac{2}{3} MR^2$, while the one of a rectangle rotated about its center is $I = \frac{1}{12} ML^2$. Applying these equations to the problem we get:

$$I = \frac{2}{3} 2400 \times 5^2 + 2 \times \left(\frac{1}{12} 1200 \times 44^2 \right) = \mathbf{427{,}200\ kgm^2}$$

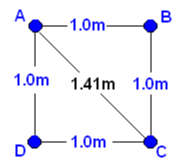
Problem 2.- Calculate the moment of inertia of four point masses arranged in a square shape shown in the figure. Take an axis of rotation that goes through point A perpendicular to the plane of the square. Each mass is 1.41 kg.



Solution: The moment of inertia of a combination of point particles can be calculated by adding the individual masses times the distance to the axis of rotation squared as follows:

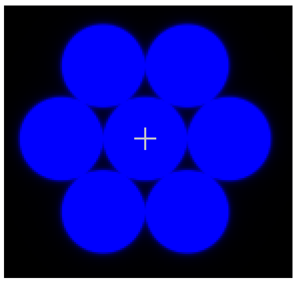
$$I = \sum m_i r_i^2$$

In the case of the masses given, only three contribute to this sum: B, C and D:



$I = 1.41kg(1m)^2 + 1.41kg(1.41m)^2 + 1.41kg(1m)^2 =$ **5.64 kgm²**

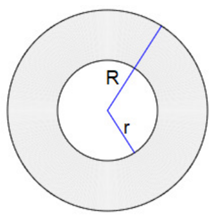
Problem 3.- Seven disks are arranged in a hexagonal pattern as shown in the figure below. Each disk has mass *m* and radius *R*. What is the moment of inertia of the system about an axis that passes through the center of the central disk and is normal to the plane?



Solution: We use the parallel axis theorem for the six disks whose centers are a distance 2*R* away from the rotation axis:

$$I = \frac{1}{2}mR^2 + 6\left(\frac{1}{2}mR^2 + 4mR^2\right) = \frac{55}{2}mR^2$$

Problem 4.- Find the moment of inertia of a washer of mass *M,* external radius *R* and internal radius r, about its center.



Solution: We can find the moment of inertia using a double integral:

$I = \int r^2 dm = \int_0^{2\pi}\int_r^R r^2 \rho r dr d\theta$, where ρ is the surface density of the disk.

Integrating we get:

$$I = 2\pi\rho\int_r^R r^3 dr = 2\pi\rho\left(\frac{R^4 - r^4}{4}\right), \text{ but } \rho = \frac{M}{\pi(R^2 - r^2)},$$

$$I = \frac{2\pi M\left(\frac{R^4 - r^4}{4}\right)}{\pi(R^2 - r^2)} = M\frac{R^2 + r^2}{2}$$

We can also consider the disk and hole as two disks with the hole having a negative mass. Then you can use the equation of a disk twice:

$$I = M\frac{R^2}{R^2 - r^2}\frac{R^2}{2} - M\frac{r^2}{R^2 - r^2}\frac{r^2}{2} = M\frac{R^2 + r^2}{2}$$

Problem 4a.- A washer has an internal radius of 0.01m and external radius of 0.02m and mass 0.004kg. Calculate the moment of inertia of the washer with respect to an axis of rotation located at its center.

Solution: We can divide the washer into infinitesimally thin rings of radius r and thickness *dr* and integrate the moment of inertia of all these rings.
Notice that the mass of the infinitesimal ring is

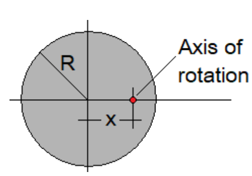
$$dm = \frac{M}{\pi(r_2^{\,2} - r_1^{\,2})}2\pi r dr$$

Where we multiply the surface density of the washer times the area of the ring. Now integrating:

$$I = \int r^2 dm = \int_{r_1}^{r_2} r^2 \frac{M}{\pi(r_2^{\,2} - r_1^{\,2})}2\pi r dr = \frac{M(r_2^{\,2} + r_1^{\,2})}{2}$$

With the values in the problem: $I = \frac{0.004(0.02^2 + 0.01^2)}{2} = \mathbf{6\times10^{-7}kgm^2}$

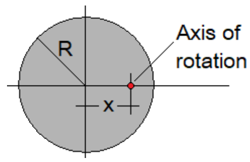
Problem 5.- A camshaft can be modeled as a disk of mass M = 1.5kg and radius R = 10cm, but whose axis of rotation is a distance *x* from the center of the disk. Calculate *x*, so the moment of inertia is twice the value with respect to the center of the disk.



Solution: The disk has a moment of inertia $\frac{1}{2}MR^2$ with respect to an axis through its center, but because it will rotate with respect to an axis parallel to this, we need to add Mx^2 to get the correct value. Then, according to the problem we get:

$$\frac{1}{2}MR^2 + Mx^2 = MR^2 \rightarrow x = \frac{\sqrt{2}}{2}R = \mathbf{7.1\ cm}$$

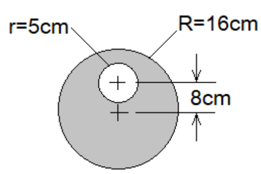
Problem 5a.- A camshaft can be modeled as a disk of mass M = 1.5kg and radius R = 10cm, but whose axis of rotation is a distance x = 6cm from the center of the disk. Calculate the moment of inertial with respect to this axis.



Solution: Using the parallel axes theorem:

$$I = \frac{1}{2}MR^2 + Mx^2 = \frac{1}{2}\times 1.5\times 0.1^2 + 1.5\times 0.06^2 = \mathbf{0.0129\ kgm^2}$$

Problem 6.- Part of a camshaft consists of a solid disk of radius 16cm, where a circle of radius 5cm has been removed as shown in the figure below. Calculate the moment of inertia with respect to an axis that goes through the center of the large circle. The mass of the object is 4kg.



Solution: We can calculate the moment of inertia of a hypothetical complete disk and subtract the one of the missing disk.
The surface density of the disk is:

$$\rho_s = \frac{M}{A} = \frac{4}{\pi 16^2 - \pi 5^2} = \frac{4}{231\pi}\frac{\text{kg}}{\text{cm}^2}$$

The masses of the complete disk and the missing one are:

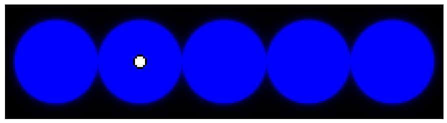
$$M_{\text{complete-disk}} = \rho_s \pi 16^2 = \frac{4\times 256}{231}\text{kg}$$

$$M_{\text{missing-disk}} = \rho_s \pi 5^2 = \frac{4\times 25}{231}\text{kg}$$

Then, the moment of inertia is:

$$I = \frac{1}{2}\frac{4\times 256}{231}\times 0.16^2 - \left(\frac{1}{2}\frac{4\times 25}{231}\times 0.05^2 + \frac{4\times 25}{231}\times 0.08^2\right) = \mathbf{0.0534\ kgm^2}$$

Problem 7.- A mechanical piece is made of five identical disks of mass m each and radius R. The disks are welded together as shown in the figure below. Calculate the moment of inertia with respect to an axis of rotation perpendicular to the plane of the disks and that passes through the center of the second one.

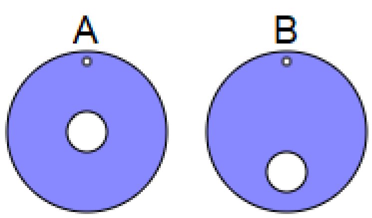


Solution: A single disk rotated about its center has moment of inertia $I = \frac{1}{2}mR^2$, but we also need to add mL^2 if rotated about an axis parallel to this. So, in this case:

$$I = \left(\frac{1}{2}mR^2 + m(2R)^2\right) + \left(\frac{1}{2}mR^2\right) + \left(\frac{1}{2}mR^2 + m(2R)^2\right) + \left(\frac{1}{2}mR^2 + m(4R)^2\right) + \left(\frac{1}{2}mR^2 + m(6R)^2\right)$$

$$I = 62.5mR^2$$

Problem 8.- Consider the disks shown in the figure. They have identical circles removed from them and are to rotate with respect to the small circles close to their border.



If the disks have the same mass, which one has larger moment of inertia with respect to the axis of rotation?

(a) *Disk A* (b) *Disk B* (c) They are the same.

Solution: We notice that the mass of A is distributed farther away from the axis of rotation, so it should have larger moment of inertia than B.
Answer: **A**