Physics I

Vectors

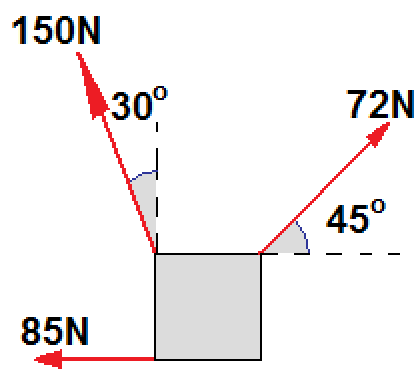
Magnitude of a vector:

In 2-D: $\left|\vec{V}\right| = \sqrt{V_x^{\ 2} + V_y^{\ 2}}$ In 3-D: $\left|\vec{V}\right| = \sqrt{V_x^{\ 2} + V_y^{\ 2} + V_z^{\ 2}}$

Angle of a vector with respect to the horizontal:

$\theta = \tan^{-1}\left(\dfrac{V_y}{V_x}\right)$ [Watch out for quadrants II and III where you need to add 180º]

Problem 1.- The figure below shows three forces acting on an object. Calculate the sum of the forces. Answer with magnitude and angle.



Solution: We write the vectors as components:

$$\vec{A} = (72\cos 45^\circ, 72\sin 45^\circ) = (50.9, 50.9)$$

$$\vec{B} = (-150\sin 30^\circ, 150\cos 30^\circ) = (-75, 130)$$

$$\vec{C} = (-85, 0)$$

Now we can add them component by component:

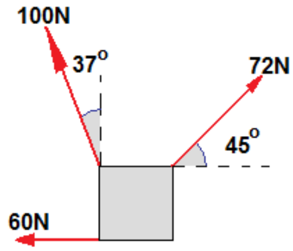
$$\vec{A} + \vec{B} + \vec{C} = (50.9 - 75 - 85, \quad 50.9 + 130 + 0) = (-109.1, \quad 180.9)$$

Finally, we write the sum in terms of the magnitude and angle:

$$\left|\vec{A} + \vec{B} + \vec{C}\right| = \sqrt{109.1^2 + 180.9^2} = \mathbf{211\ N}$$

$$\theta = \tan^{-1}\left(\frac{180.9}{-109.1}\right) = -58.9 + 180 = \mathbf{121^\circ}$$

Problem 1a.- Calculate the sum of the three forces. Answer with magnitude and angle.



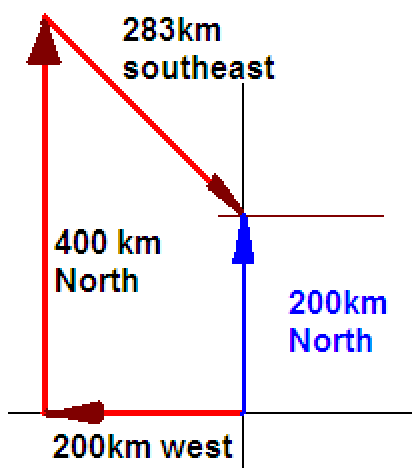
Solution:

$$
\begin{aligned}
&(72\cos 45, 72\sin 45) + \\
&(100\cos 127, 100\sin 127) + \\
&(-60, 0) \\
&= (-69.3, 130.8)
\end{aligned}
$$

=**148 N at 118°.**

Problem 2.- A car is driven 200 km west, then 400 km north and finally 283 km southeast. Draw a vector diagram "head to tail" and find the displacement of the car in km.

Solution: A carefully drawn diagram will help us solve this problem:



So, the answer is **200 km North**

It is also possible to write down the components of the vectors and add them mathematically, but the "head to tail method" is as effective in this case.

Problem 3.- Transversal velocity is the name given by astronomers to the motion of a star against the background of other farther stars. It can be measured by careful observation over long periods. Radial velocity is the velocity of the star along our line of sight, and it can be calculated by measuring the "red-shift" or "blue-shift" of its light.

Sirius, the brightest star in the night sky, has a transversal velocity of 16 km/s and a radial velocity of -8 km/s (minus sign because it is going away from us). Find the magnitude of its velocity.

Solution: The two components of the velocity form an angle of 90º, so the magnitude of the vector is:

$$\left|\vec{V}\right| = \sqrt{V_{radial}{}^{2} + V_{transversal}{}^{2}} = \sqrt{16^2 + 8^2}\,km/s = \mathbf{17.9\ m/s}$$

Problem 4.- What is the y component of a vector (in the xy plane) whose magnitude is 250 m and whose x component is 240m?

Solution: Graphically we know that the problem will have two solutions. Mathematically we know that since the magnitude is 250 m.

$\sqrt{x^2 + y^2} = 250$, and x = 240, so

$\sqrt{240^2 + y^2} = 250 \rightarrow y^2 = 250^2 - 240^2 \rightarrow y^2 = 4900 \rightarrow$ **y=70 m or y= -70m**

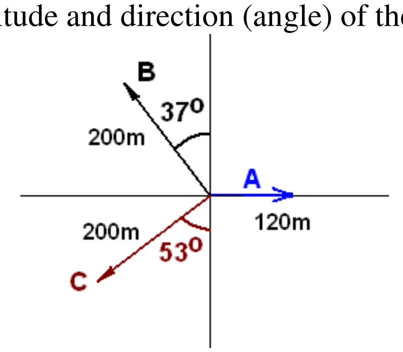
Problem 4a.- What is the y component of a vector (in the xy plane) whose magnitude is 410 m and whose x component is 400m?

Solution: Graphically we know that the problem will have two solutions. Mathematically we know that since the magnitude is 410 m.

$\sqrt{x^2 + y^2} = 410$, and x=400, so:

$\sqrt{400^2 + y^2} = 410 \rightarrow 160000 + y^2 = 168100 \rightarrow y^2 = 8100 \rightarrow$ **y=90 m or y= -90m**

Problem 5.- Calculate the magnitude and direction (angle) of the vector $\vec{A} + \vec{B} + \vec{C}$:



Solution: Let us represent the vectors as components:

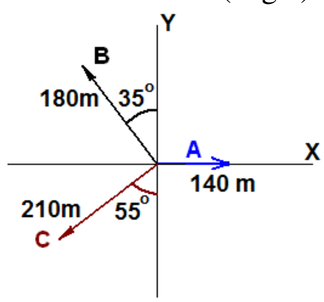
$$A = (120{,}0)$$
$$B = (-200\sin(37^o),\ 200\cos(37^o)) = (-120,\ 160)$$
$$C = (-200\sin(53^o),\ 200\cos(53^o)) = (-160,\ -120)$$

To get $\vec{A}+\vec{B}+\vec{C}$ we add the components of A, B and C:

$$\vec{A}+\vec{B}+\vec{C} = (120-120-160,\ 0+160-120) = (\text{-}160,\ 40)$$

The magnitude of this vector is: $\sqrt{160^2+40^2}$ = **165 m.**
And the angle:
$\tan^{-1}\left(\frac{40}{-160}\right)$ = -14º, but we add 180º because it is in the second quadrant: **166º**

Problem 6a.- Calculate the magnitude and direction (angle) of the vector $\vec{A}+\vec{B}+\vec{C}$:



Solution: Let us represent the vectors as components:

$$A = (140{,}0)$$
$$B = (180\cos 125^o,\ 180\sin 125^o) = (-103.2,\ 147.4)$$
$$C = (210\cos 215^o,\ 210\sin 215^o) = (-172,\ -120)$$

To get $\vec{A}+\vec{B}+\vec{C}$ we add the components of A, B and C:

$$\vec{A}+\vec{B}+\vec{C} = (140-103.2-172,\ 0+147.4-120) = (\text{-}135.2,\ 27.4)$$

The magnitude of this vector is: $\sqrt{135.2^2+27.4^2}$ = **138 m.**
And the angle:
$\tan^{-1}\left(\frac{27.4}{-135.2}\right)$ = -11.5º, but we add 180º because it is in the second quadrant: **168.5º**

Problem 7.- A plumber steps out of his truck, walks 25 m east and 50 m south, and then takes an elevator 11 m down into the basement of a building where a bad leak is occurring. What is the magnitude of his displacement?

Solution: If we take east to be the x-axis, north to be the y-axis and up to be the z-axis we have a displacement of **(25 m, -50 m, -11 m)** whose magnitude is:

$$\sqrt{(25\text{m})^2 + (50\text{m})^2 + (11\text{m})^2} = \mathbf{57\ m}$$

Problem 8.- A car is driven 200 km west and then 141 km southeast. What is the displacement of the car from the point of origin (magnitude and direction)? Draw a diagram.

Solution: If we select North to be the y-axis and East to be the x-axis, the first leg of the trip will be (-200 km, 0) and the second leg (141 km cos(45º), -141 km sin(45º)), so the total displacement is:

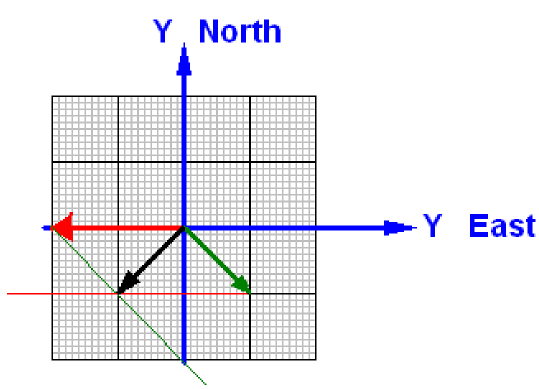
(-200 km , 0)

(141 km cos45º, -141 km sin45º)

(-100 km , -100 km)

The magnitude is $\sqrt{(100\text{km})^2 + (100\text{km})^2}$ = **141 km.**

Direction: $\theta = \tan^{-1}\left(\frac{-100}{-100}\right)$=45º, but since the vector is in the third quadrant, we add 180º, getting **225º**



In the diagram, the first vector is in red, the second in green and the resultant in black.

Problem 9.- A map shows that the summit of a mountain is 635m north and 845m west of your location (base camp). If the summit is 2,210 m higher than base camp, what is the straight distance between base camp and the summit?

Solution: This is a problem of finding the magnitude of a three-dimensional vector, which can be calculated with:

$$\left|\vec{R}\right| = \sqrt{x^2 + y^2 + z^2} = \sqrt{317^2 + 423^2 + 1105^2} = \mathbf{1225\ m}$$

Problem 10.- Calculate the angle between the vectors $\vec{A} = 3i + 4j$ and $\vec{B} = -i + j$

Solution: Vector A is in the first quadrant and its angle is $\theta_A = \tan^{-1}\left(\frac{4}{3}\right) = 53^\circ$

Vector B is in the second quadrant and its angle is $\theta_B = \tan^{-1}\left(\frac{1}{-1}\right) + 180 = 135^\circ$

The difference is **82º.**

Problem 11.- Calculate the angle between the vectors $\vec{A} = (1, 1, 1)$ and $\vec{B} = (1, -1, 2)$

Solution:

$$\vec{A} \cdot \vec{B} = A_x B_x + A_y B_y + A_z B_z = \left|\vec{A}\right|\left|\vec{B}\right| \cos \angle_A^B$$

$$\rightarrow 1 \times 1 + 1 \times (-1) + 1 \times 2 = 2 = \sqrt{1^2 + 1^2 + 1^2}\sqrt{1^2 + 1^2 + 2^2}\, Cos \angle_A^B$$

$$\rightarrow \cos \angle_A^B = \frac{2}{\sqrt{3}\sqrt{6}} \rightarrow \angle_A^B = \cos^{-1}\left(\frac{\sqrt{2}}{3}\right) = 62^\circ$$

Problem 12.- A delivery truck follows a route that goes north for 5.0 miles, then east for 10.0 miles and finally north-east for 14.2 miles. Calculate the magnitude and direction (angle) of the total displacement.

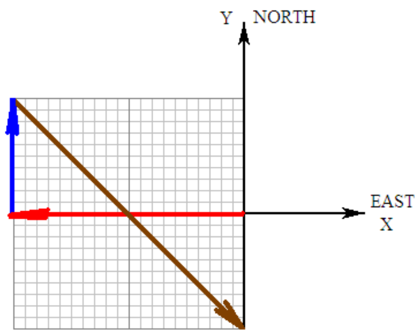
Solution: If we select North to be the y-axis and East to be the x-axis, the first leg of the trip will be (0, 5), the second leg (10, 0) and the third (14.2cos45º, 14.2sin45º)=(10, 10), so the total displacement is: (20,15) giving a magnitude of

The magnitude of this vector is: $\sqrt{20^2 + 15^2} =$ **25 miles.**
And the angle:

$$\tan^{-1}\left(\frac{15}{20}\right) = \mathbf{37^o}$$

Problem 13.- A car is driven 200 km west, then 100 km north and finally 283 km southeast. Draw a diagram and find the displacement of the car from the point of origin (magnitude and direction).

Solution: A diagram of the displacement would look as follows:



The vectors:
$V_1 = (-200km, 0)$
$V_2 = (0, 100km)$
$V_3 = (283km Cos 45, -283km \sin 45) = (200km, -200km)$
So: $V_1 + V_2 + V_3 = (0, -100km)$
The final displacement will be **100km to the south**.

Problem 14.- A car is driven 200 km west and then 141 km southeast. What is the displacement of the car from the point of origin (magnitude and direction)? Draw a diagram.

Solution: If we select North to be the y-axis and East to be the x-axis, the first leg of the trip will be (-200 km, 0) and the second leg (144 km cos(45º), -144 km sin(45º)). Refer to the figure to determine the angles and signs, so the total displacement is:

(-200 km, 0) + (144 km cos(45º), -144 km sin(45º)) = (-98 km, -102 km)

The magnitude is $\sqrt{(98\text{km})^2 + (102\text{km})^2}$ = **141 km.**

Direction: $\theta = \tan^{-1}\left(\frac{-102}{-98}\right)$=46.1º, but since the vector is in the third quadrant, we add 180º, getting **226.1º**