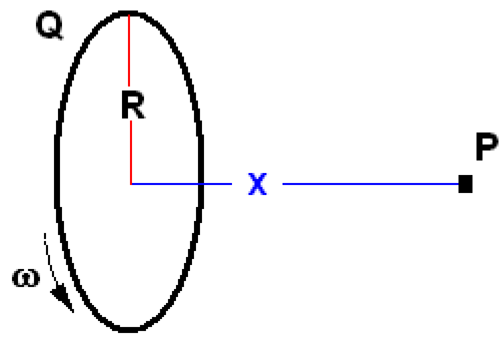
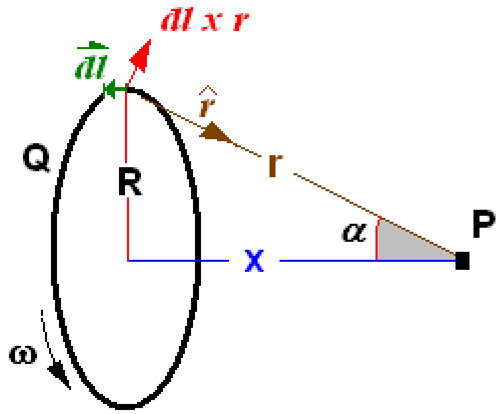
Physics II

Rotating Charged Ring and Disk

Problem 1.- Find the magnetic field at point P due to a ring of radius R, uniformly charged with charge Q rotating with angular frequency ω as shown in the figure.



Solution: We use Biot and Savart's law to find the magnetic field. Let's divide the ring into differentials of length *dl* as follows:



In Biot and Savart's law: $d\vec{B} = \frac{\mu_\circ}{4\pi}\frac{Id\vec{l}\times\hat{r}}{r^2}$ the electric current is the charge divided by time, in this case it is Q divided by one period of rotation, which can be written in terms of the angular velocity:

$$I = \frac{Q}{T} = Qf = \frac{Q\omega}{2\pi}$$

Notice that the angle between the vectors $d\vec{l}$ and $\hat{r}$ is 90º, so the cross product will have magnitude equal to $d\vec{l}$ (since the magnitude of $\hat{r}$ is 1) and it will be pointed at 90º to the plane formed by the two vectors.

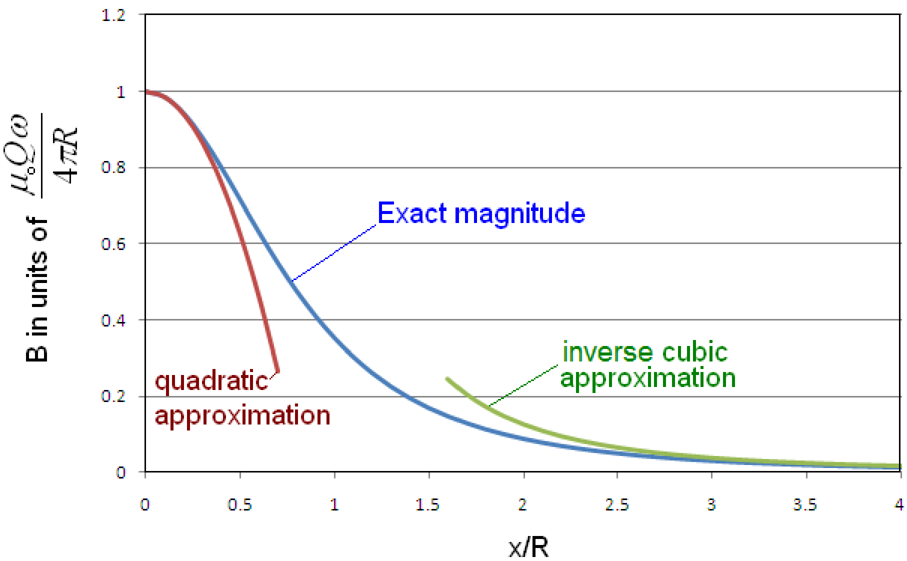
Different vectors $d\vec{l}$ and $\hat{r}$ will generate cross products in different directions and because of symmetry only the x-component will remain after integrating, so we only need to consider the x-component. The projection of $d\vec{l}\times\hat{r}$ on the x-axis is $dl\sin\alpha$, so the integral will be:

$$B = \int \frac{\mu}{4\pi} \frac{Id\vec{l} \times \hat{r}}{r^2} = \frac{\mu_\circ}{4\pi} \frac{I \sin\alpha}{r^2} \int dl = \frac{\mu_\circ}{4\pi} \frac{I \sin\alpha}{r^2} (2\pi R) = \frac{\mu_\circ}{2} \frac{IR \sin\alpha}{r^2}$$

Replacing $\sin\alpha = \frac{R}{r}$, and $r = \sqrt{R^2 + x^2}$ we get:

$$B = \frac{\mu_\circ}{2} \frac{IR^2}{(R^2 + x^2)^{3/2}} \text{ or } B = \frac{\mu_\circ}{4\pi} \frac{QR^2\omega}{(R^2 + x^2)^{3/2}}$$

A graph in terms of x/R looks as follows: $B = \frac{\mu_\circ Q\omega}{4\pi R} \frac{1}{\left[1 + (x/R)^2\right]^{3/2}}$



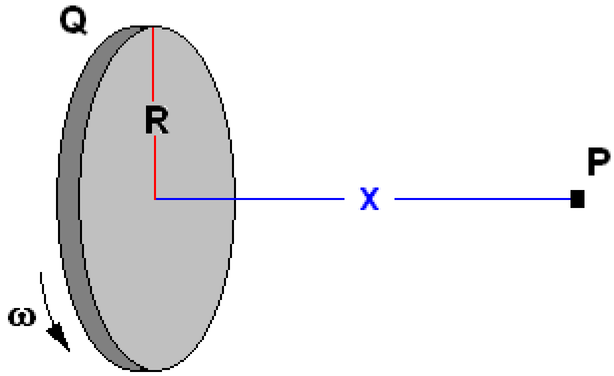
Notice that for values of x <<R we can approximate the value of B to:

$B \approx \frac{\mu_\circ Q\omega}{4\pi R}\left[1 - \frac{3x^2}{2R^2}\right]$, which is a parabolic approximation.

And for values of x>>R we can approximate the value of B to:

$B = \frac{\mu_\circ Q\omega R^2}{4\pi x^3}$, which is an inverse cubic approximation.

Problem 2.- Find the magnetic field at point P due to the disk of radius R, uniformly charged with charge Q rotating with angular frequency of ω as shown in the figure.



Solution: In problems like this, we want to divide the object (the disk) into simpler figures, like rings. Since we already have a solution for a ring of charge Q rotating with angular velocity ω we write the same equation in terms of a differential of charge:

$$dB = \frac{\mu_\circ}{4\pi}\frac{r^2\omega}{(r^2+x^2)^{3/2}}dQ$$

If we consider a thin ring the differential of charge will be: $dQ = \frac{Q}{\pi R^2}2\pi r dr = \frac{Q2rdr}{R^2}$ and substituting this in the equation above, we get:

$$dB = \frac{\mu_\circ}{4\pi}\frac{r^2\omega}{(r^2+x^2)^{3/2}}\frac{Q2rdr}{R^2} = \frac{\mu_\circ \omega Q}{2\pi R^2}\frac{r^3 dr}{(r^2+x^2)^{3/2}}$$

To integrate we have several alternatives. For example, by substitution: $r = x\tan\theta$

$$B = \frac{\mu_\circ \omega Q}{2\pi R^2}\int_0^R \frac{r^3 dr}{(r^2+x^2)^{3/2}} = \frac{\mu_\circ \omega Q}{2\pi R^2}\int_0^{\tan^{-1}(R/x)} \frac{x^3\tan^3\theta x\sec^2\theta d\theta}{x^3\sec^3\theta} = \frac{\mu_\circ \omega Q x}{2\pi R^2}\int_0^{\tan^{-1}(R/x)} \frac{\sin^3\theta d\theta}{\cos^2\theta}$$

$$B = -\frac{\mu_\circ \omega Q x}{2\pi R^2}\int_0^{\tan^{-1}(R/x)} \frac{(1-\cos^2\theta)d\cos\theta}{\cos^2\theta} = \frac{\mu_\circ \omega Q x}{2\pi R^2}\left[\frac{1}{\cos\theta}+\cos\theta\right]_0^{\tan^{-1}(R/x)}$$

$$B = \frac{\mu_\circ \omega Q x}{2\pi R^2}\left[\frac{\sqrt{x^2+R^2}}{x}+\frac{x}{\sqrt{x^2+R^2}}-2\right]$$