Quantum Mechanics

Expectation values

Problem 1.- Considering the wave function at time zero:

$$\psi(x,0) = \begin{cases} c(a^2 - x^2) & if \; -a < x < a \\ 0 & elsewhere \end{cases}$$

a) Find the normalization constant c.
b) Find the expectation value of x.
c) Find the expectation value of p.
d) Find the expectation value of x^2.
e) Find the expectation value of p^2.
f) Find the uncertainty in x.
g) Find the uncertainty in p.
h) Verify that the uncertainties of x and p satisfy the uncertainty principle.

Solution:

a) To find the normalization constant we need to make sure that: $\int_{-\infty}^{\infty} \psi^* \psi dx = 1$, so:

$$\int_{-a}^{a} \left[c(a^2 - x^2)\right]^2 dx = 1 \rightarrow \int_{-a}^{a} c^2(a^4 - 2a^2x^2 + x^4)dx = 1 \rightarrow c^2(a^4x - 2a^2x^3/3 + x^5/5)\Big|_{-a}^{a} = 1$$

$$2c^2(a^5 - 2a^5/3 + a^5/5)\Big| = 1 \rightarrow \frac{16c^2a^5}{15} = 1 \rightarrow c = \sqrt{\frac{15}{16a^5}}$$

b) To get the expectation value of x we compute the integral:

$$\langle x \rangle = \int_{-\infty}^{\infty} \psi^* x \psi dx$$

$$\langle x \rangle = \int_{-a}^{a} x\left[c(a^2 - x^2)\right]^2 dx = 0 \rightarrow \langle x \rangle = 0$$

Notice that the integral is zero because the integrand is odd.

c) To find the expectation value of p we cannot just take the derivative of $\langle p \rangle$ because we only know its value at one point in time (t=0) instead, we use the prescription:

$$\langle p \rangle = \int_{-\infty}^{\infty} \psi^* \left(-i\hbar \frac{\partial}{\partial x}\right) \psi dx$$

With the wave function of the problem:

$$\langle p \rangle = \int_{-a}^{a} c(a^2 - x^2)\left(-i\hbar \frac{\partial}{\partial x}\right) c(a^2 - x^2)dx = \int_{-a}^{a} c(a^2 - x^2)(-i\hbar)c(-2x)dx = 0$$

The integral is zero because the integrand is an odd function. $\langle p \rangle = 0$

d) The expectation value of x^2:

$$\langle x^2\rangle = \int_{-\infty}^{\infty}\psi^* x^2\psi dx$$

$$\langle x^2\rangle = \int_{-a}^{a} x^2\left[c(a^2-x^2)\right]^2 dx = \int_{-a}^{a} x^2\left[c^2(a^4-2a^2x^2+x^4)\right]dx = \int_{-a}^{a}\left[c^2(a^4x^2-2a^2x^4+x^6)\right]dx$$

$$\langle x^2\rangle = c^2(a^4x^3/3-2a^2x^5/5+x^7/7)\Big|_{-a}^{a} = 2c^2(a^7/3-2a^7/5+a^7/7) = 2c^2a^7(1/3-2/5+1/7)$$

$$\langle x^2\rangle = 2c^2a^7\left(\frac{35-42+15}{105}\right) = 2c^2a^7(8/105) = 16c^2a^7/105$$

With the value of c calculated above:

$$\langle x^2\rangle = 16\frac{15}{16a^5}a^7/105 \rightarrow \langle x^2\rangle = \frac{a^2}{7}$$

e) Expectation value of p2:

$$\langle p^2\rangle = \int_{-\infty}^{\infty}\psi^*\left(-i\hbar\frac{\partial}{\partial x}\right)^2\psi dx = \int_{-\infty}^{\infty}\psi^*\left(-\hbar^2\frac{\partial^2}{\partial x^2}\right)\psi dx$$

With the wave function of the problem:

$$\langle p^2\rangle = \int_{-a}^{a} c(a^2-x^2)\left(-\hbar^2\frac{\partial^2}{\partial x^2}\right)c(a^2-x^2)dx = \int_{-a}^{a} c^2(a^2-x^2)\left(-\hbar^2\right)(-2)dx = 2\hbar^2c^2\int_{-a}^{a}(a^2-x^2)dx$$

$$\langle p^2\rangle = 2\hbar^2c^2(a^2x-x^3/3)\Big|_{-a}^{a} = 4\hbar^2c^2(a^3-a^3/3) = 8\hbar^2c^2a^3/3 = 8\hbar^2\left(\sqrt{\frac{15}{16a^5}}\right)^2 a^3/3$$

$$\langle p^2\rangle = \frac{5\hbar^2}{2a^2}$$

f) Uncertainty in x: $\sigma_x = \sqrt{\langle x^2\rangle - \langle x\rangle^2} = \sqrt{\frac{a^2}{7}} \rightarrow \quad \sigma_x = \frac{\sqrt{7}a}{7}$

g) Uncertainty in p: $\sigma_p = \sqrt{\langle p^2\rangle - \langle p\rangle^2} = \sqrt{\frac{5\hbar^2}{2a^2}} \rightarrow \quad \sigma_p = \frac{\sqrt{10}\hbar}{2a}$

h) Verifying the uncertainty principle: $\sigma_p\sigma_x = \frac{\sqrt{10}\hbar}{2a}\frac{\sqrt{7}a}{7} = \frac{\sqrt{70}\hbar}{14} \approx 0.59\hbar > \frac{\hbar}{2}$