Quantum Mechanics

Finite Square Well Potential

Problem 1.- Find the odd wave functions of the finite square well potential. Pay special attention to the very shallow and the very narrow cases.

Solution: *d bound states of the finite square well potential.*

A finite square potential of depth -V_o and length 2a will have the function:

$$\psi(x) = \begin{cases} -Ae^{kx} & for \quad x < -a \\ C\sin lx & for \quad -a < x < a \\ Ae^{-kx} & for \quad x > a \end{cases}$$

Where: $k = \frac{\sqrt{-2mE}}{\hbar}$ and $l = \frac{\sqrt{2m(V_o + E)}}{\hbar}$

Continuity at x = a implies that:

$$Ae^{-ka} = C\sin la$$

And continuity of the first derivative at x=a means that:

$$-Ake^{-ka} = Cl\cos la$$

Dividing these equations, we get:

$$-k = l\cot la \qquad (*)$$

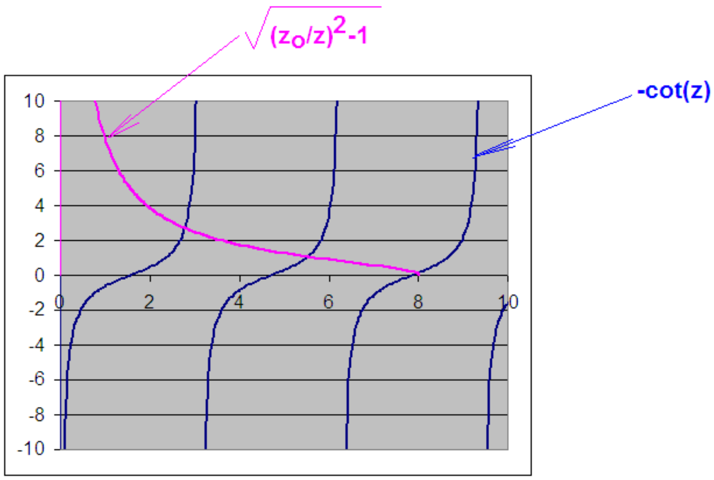
To solve the problem graphically we change variables:

$z = la$ and $z_o = \frac{a}{\hbar}\sqrt{2mV_o}$

With these changes: $ka = \sqrt{{z_o}^2 - z^2}$, so the equation (*) can be written as:

$$-\sqrt{{z_o}^2 - z^2} = z\cot z \rightarrow \sqrt{\left(\frac{z_o}{z}\right)^2 - 1^2} = -\cot z$$

A graph for the case $z_o = 8$ is shown in the figure. Notice that in that case there are three solutions corresponding to three intersections.



Let us now analyze the two limiting cases:

a) *Wide deep well*: Here the value of z_o is very large, and the solutions will be all close to even multiples of π, so z=$2n\pi$, meaning that :

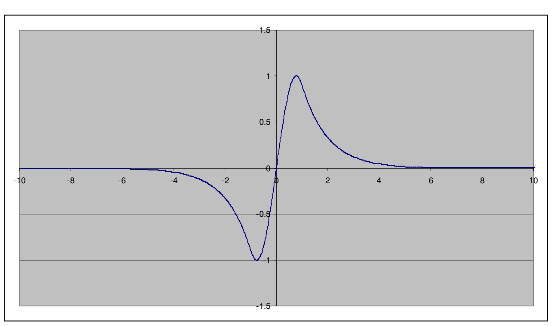
$$\frac{2n\pi}{a} = \frac{\sqrt{2m(V_o + E)}}{\hbar} \rightarrow V_o + E = \frac{n^2\pi^2\hbar^2}{2ma^2}$$

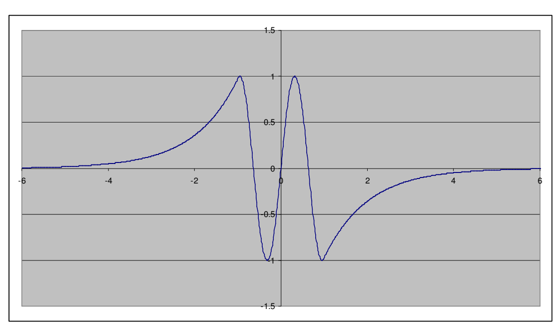
These are precisely the odd solutions of the infinite square well potential.

b) *Shallow narrow well:* in the even case, we found that no matter how shallow the potential well is, there is always at least one bound state. This is not the case for odd solutions, because the minimum value of z_o to get at least one intersection is $\pi/2$, so the minimum potential depth is:

$$\pi/2 = \frac{a}{\hbar}\sqrt{2mV_o} \rightarrow V_o = \frac{\pi^2\hbar^2}{8ma^2}$$

Here are some odd solutions:





Problem 2.- What would be the minimum depth of a one-dimensional finite square well if we require at least one odd bound state for electrons confined in a 15nm trap?

Solution: No matter how shallow a potential well is in one dimension, you can always find at least one even bound state, but for odd states you need at least $Z_o = \frac{\pi}{2}$, which means:

$$\pi/2 = \frac{a}{\hbar}\sqrt{2mV_o} \rightarrow V_o = \frac{\pi^2\hbar^2}{8ma^2}$$

Given that a=15nm, we get:

$$V_o = \frac{\pi^2\hbar^2}{8ma^2} = \frac{(3.1416)^2(1.05\times10^{-34}\,Js)^2}{8(9.1\times10^{-31}\,kg)(15\times10^{-9}\,m)a^2} = \mathbf{6.64\times10^{-23}\ J}$$