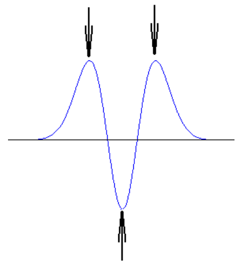
Quantum Mechanics

Harmonic oscillator

Problem 1.- Sketch the eigen function of the *second excited state* of the harmonic oscillator and indicate in your drawing the points of maximum probability of finding the particle.

Solution: A sketch of the eigen function of to the *second excited state* of the harmonic oscillator:



The arrows indicate the points of maximum probability of finding the particle.

Problem 2.- The functions $|0\rangle, |1\rangle$ and $|2\rangle$ are normalized solutions of the harmonic oscillator problem with quantum numbers n=0, 1 and 2. Calculate the expectation value of the energy of the state:

$$|\psi\rangle = \frac{2|0\rangle + 3|1\rangle + |2\rangle}{\sqrt{14}}$$

Solution: Recall that the Hamiltonian for the harmonic oscillator is:

$H = \hbar\omega(a^+ a^- + 1/2)$,

which applied to an eigen function $|n\rangle$ yields:

$H|n\rangle = (n + 1/2)\hbar\omega|n\rangle$, so, the expectation value of the energy is:

$$E = \langle\psi|H|\psi\rangle = \left(\frac{2\langle 0| + 3\langle 1| + \langle 2|}{\sqrt{14}}\right) H \left(\frac{2|0\rangle + 3|1\rangle + |2\rangle}{\sqrt{14}}\right)$$

$$E = \left(\frac{2\langle 0| + 3\langle 1| + \langle 2|}{\sqrt{14}}\right)\left(\frac{2(1/2\omega\hbar)|0\rangle + 3(3/2\omega\hbar)|1\rangle + (5/2\omega\hbar)|2\rangle}{\sqrt{14}}\right)$$

The eigen functions are orthogonal to each other, so only same functions will produce nonzero internal products:

$$E = \frac{4(1/2\omega\hbar)\langle 0\|0\rangle + 9(3/2\omega\hbar)\langle 1\|1\rangle + (5/2\omega\hbar)\langle 2\|2\rangle}{14}$$

Since the eigen functions are already normalized, the internal products are equal to 1:

$$E = \frac{4(1/2\omega\hbar) + 9(3/2\omega\hbar) + (5/2\omega\hbar)}{14} = \frac{9}{7}\hbar\omega$$

Problem 3.- What is the expectation value of the energy E, for a particle in the harmonic oscillator potential with wave function:

$$\psi = 0.6|0\rangle + 0.8|1\rangle$$

Solution: Recall that the expectation value is equal to the sum of the eigenvalues multiplied by their probabilities:

$$\langle E\rangle = 0.6^2(1/2\hbar\omega) + 0.8^2(3/2\hbar\omega) = 1.14\hbar\omega$$

Problem 4.- What is the expectation value of x^2 for a particle in the harmonic oscillator potential with the wave function $\psi = 0.6|0\rangle + 0.8|1\rangle$ **?**

Solution: Let us find the expectation value of x^2:

$$\langle x^2\rangle = (0.6\langle 0| + 0.8\langle 1|)x^2(0.6|0\rangle + 0.8|1\rangle) = \frac{\hbar}{2m\omega}(0.6\langle 0| + 0.8\langle 1|)(a^+ + a^-)^2(0.6|0\rangle + 0.8|1\rangle)$$

We consider only the terms a^+a^- and a^-a^+ in the expectation value. The other terms will yield $|2\rangle, |3\rangle$ and zero, which do not contribute.

$$\langle x^2\rangle = \frac{\hbar}{2m\omega}(0.6^2(1) + 0.8^2(3)) = 1.14\frac{\hbar}{m\omega}$$

Problem 5.- What is the expectation value of p^2 in a harmonic oscillator with wave function $|3\rangle$?

Solution: Writing p as an operator:

$$p = i\sqrt{\frac{m\omega\hbar}{2}}\left(a^+ - a^-\right)$$

The expectation value will be:

$$\langle 3|\left(i\sqrt{\frac{m\omega\hbar}{2}}\left(a^+ - a^-\right)\right)^2|3\rangle = -\frac{m\omega\hbar}{2}\langle 3|\left(a^+ - a^-\right)^2|3\rangle = -\frac{m\omega\hbar}{2}\langle 3|\left(a^{+2} - a^+a^- - a^-a^+ + a^{-2}\right)|3\rangle$$

Using the rules: $a^+|n\rangle = \sqrt{n+1}|n+1\rangle$ and $a^-|n\rangle = \sqrt{n}|n-1\rangle$ we get:

$$= \frac{m\omega\hbar}{2}\langle 3|\left(a^+a^- + a^-a^+\right)|3\rangle = \frac{7m\omega\hbar}{2}$$