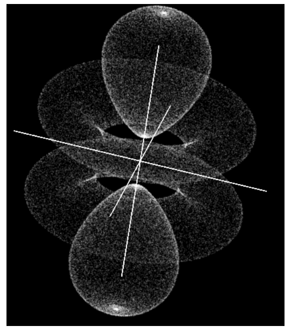
Quantum Mechanics

Hydrogen atom



Problem 1.- Write down the wave function of the hydrogen atom with quantum numbers n=2, l=1 and m=1 and indicate where the probability of finding the electron vanishes.

Solution: $\psi_{211} = -\sqrt{\frac{3}{8\pi}} \sin\theta e^{i\phi} \frac{1}{\sqrt{24}} a^{-3/2} \frac{r}{a} e^{-r/2a}$

The probability of finding the electron vanishes when $\theta = 0$, $\theta = \pi$ and r=0.

Problem 2.- Calculate the mean value of $\frac{1}{\text{r}}$ in the ground state of the hydrogen atom.\

Solution: We calculate the mean value of $\frac{1}{\text{r}}$ by integrating the square of the wave function times $\frac{1}{\text{r}}$ over all space:

$$\left\langle \frac{1}{r} \right\rangle = \int_0^\infty \frac{1}{\pi a^3} \frac{1}{r} e^{-2r/a} 4\pi r^2 dr = \frac{4}{a^3} \int_0^\infty e^{-2r/a} r dr = \frac{4}{a^3} \frac{a^2}{4} \int_0^\infty e^{-2r/a} 2r/a d2r/a$$

With the change of variable: $y = 2r/a$ we get:

$$\left\langle \frac{1}{r} \right\rangle = \frac{1}{a} \int_0^\infty e^{-y} y dy$$

integrating by parts: $\left\langle \frac{1}{r} \right\rangle = \frac{1}{a} \left(-ye^{-y} \Big|_0^\infty + \int_0^\infty e^{-y} dy \right) = \frac{1}{a}$

Problem 3.- Calculate the probability of a hydrogen atom in the ground state with the electron closer than “r” to the nucleus. The value of “a” is the Bohr radius (0.529 Å)

Solution:

$$P = \int_0^r |\psi|^2 4\pi r^2 dr = \int_0^r \frac{1}{\pi a^3} e^{-2r/a} 4\pi r^2 dr = \frac{1}{2} \int_0^{2r/a} e^{-u} u^2 du = -\frac{1}{2} \int_0^{2r/a} u^2 de^{-u} =$$

$$P=-\frac{1}{2}\left[u^2e^{-u}\Big|_0^{2r/a}+\int_0^r 2ude^{-u}\right]=-\frac{1}{2}\left[u^2e^{-u}+2ue^{-u}\Big|_0^r+\int_0^r 2de^{-u}\right]$$

$$P=-\frac{1}{2}\left[u^2e^{-u}+2ue^{-u}+2e^{-u}\right]_0^{2r/a}=1-\left(\frac{2r^2}{a^2}+\frac{2r}{a}+1\right)e^{-2r/a}$$

Problem 4.- Consider the reduced mass to solve the problems.

A) Find the correction to the error in the energy of the hydrogen atom by not considering the motion of the nucleus
B) Find the difference in wavelength between the red Balmer line in deuterium and hydrogen
C) Find the binding energy of positronium (like hydrogen, but with a positron instead of a proton).
D) Find the Lyman-alpha wavelength for muonic hydrogen, where the electrons is replaced by a muon, 206.77 times heavier than the electron.

Solution:

A) $E_{1-approximation}=-\left[\frac{m}{2\hbar^2}\left(\frac{e^2}{4\pi\varepsilon_\circ}\right)^2\right]$

$E_{1-exact}=-\left[\frac{\mu}{2\hbar^2}\left(\frac{e^2}{4\pi\varepsilon_\circ}\right)^2\right]$

Error: $\frac{E_{1-approximation}-E_{1-exact}}{E_{1-exact}}\times 100\%=\frac{m-\mu}{\mu}\times 100\%=\frac{m-\frac{mM}{m+M}}{\frac{mM}{m+M}}\times 100\%=\frac{m}{M}\times 100\%$ =0.054%

B) Balmer wavelength in D is shorter than H by:

$$\frac{\mu_H}{\mu_D}=\frac{\frac{mM_P}{m+M_P}}{\frac{mM_D}{m+M_D}}=\frac{M_P}{M_D}\frac{m+M_D}{m+M_P}=\frac{1}{2}\frac{m+2M_P}{m+M_P}=0.9997278$$

In wavelength: $(1-0.9997278)\times 656.3nm=0.18nm$

C) 6.8eV

D) $$\frac{\mu_{muonic}}{\mu_H}=\frac{\frac{m_{muon}M_P}{m_{muon}+M_P}}{\frac{mM_P}{m+M_P}}=\frac{m_{muon}}{m}\frac{m+M_P}{m_{muon}+M_P}=\frac{206.77m}{m}\frac{m+1836.15m}{206.77m+1836.15m}=185.9$$

Wavelength: $\frac{\lambda_H}{\lambda_{muonic}}=185.9\rightarrow\lambda_{muonic}=\frac{\lambda_H}{185.9}=\frac{121.5nm}{185.9}=0.65nm$