Quantum Mechanics

Spin

Problem 1.-Find the value of "A" that normalizes the spinor χ and calculate the probability of measuring S_Z to be $+\hbar/2$.

$$\chi = A\begin{pmatrix} 1+i \\ 1-i \end{pmatrix}$$

Solution: To normalize the spinor, we need to make sure that the sum of the squares of the absolute value of the coefficients is equal to one:

$$\chi = A\begin{pmatrix} 1+i \\ 1-i \end{pmatrix} \qquad |A(1-i)|^2 + |A(1+i)|^2 = 1 \rightarrow |A|^2(1+1+1+1) = 1 \rightarrow |A|^2 = \frac{1}{4}$$

"A" can be a complex number, so in general it could be $\frac{1}{2}e^{i\alpha}$ for any real α, but we can choose to make A=$\frac{1}{2}$

Notice that we can write the spinor as a linear combination of the eigen functions of S_z:

$$\chi = A\begin{pmatrix} 1-i \\ 1+i \end{pmatrix} = A(1-i)\begin{pmatrix} 1 \\ 0 \end{pmatrix} + A(1+i)\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

The only possible results are the eigenvalues $\hbar/2$ and $-\hbar/2$, so the probabilities are the coefficients squared:

Probability of $S_Z = \hbar/2 = |A(1-i)|^2 = \frac{1+1}{4} = 1/2$

Problem 2.- In the spinor from the previous problem, what is the probability that a measurement of S_X will yield $+\hbar/2$?

Solution: We need to write the spinor as a linear combination of eigenfunctions of S_x:

$$\chi = \begin{pmatrix} (1-i)/2 \\ (1+i)/2 \end{pmatrix} = X\begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} + Y\begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}$$

This gives us two simultaneous linear equations:

$$(1-i)/2 = X/\sqrt{2} + Y/\sqrt{2}$$
$$(1+i)/2 = X/\sqrt{2} - Y/\sqrt{2}$$

We can add the equations to get X:

$$(1-i)/2+(1+i)/2 = X\sqrt{2} \rightarrow X = \frac{2}{2\sqrt{2}} = \frac{1}{\sqrt{2}}$$

The square of the absolute value of X will be the probability of getting $S_X = \hbar/2$:

Probability of $S_X = \hbar/2 = \frac{1}{2}$

Problem 3.- What is the angle between the "z" axis and the spin of an electron in the state $\begin{pmatrix}1\\0\end{pmatrix}$?

Solution: The projection of the vector on the z-axis is $\hbar/2$ and the magnitude of the vector is $\sqrt{S^2} = \sqrt{\frac{\hbar}{2}\left(\frac{3\hbar}{2}\right)} = \frac{\sqrt{3}}{2}\hbar$ so the angle is:

$$\theta = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right) = \mathbf{54.7^o}$$

Problem 3a.- What is the angle between the "z" axis and the spin of an iron cluster in the state $|S, S_Z\rangle = |100, 100\rangle$ Ignore any orbital angular momentum.

Solution: The projection of S on the z axis is $100\hbar$ and the absolute value of the spin is $\sqrt{S^2} = \sqrt{100(101)\hbar^2}$ so:

$$\theta = \cos^{-1}\left(\frac{100}{\sqrt{100(101)}}\right) = \mathbf{5.7^o}$$

Problem 4.- A Stern-Gerlach apparatus measures S_Z of an electron to be $+\hbar/2$. What will be the outcome of a measurement of S_X made just after that?

Solution: Since the value of S_Z is known, there is no information about S_X.
A measurement of S_X will give $\hbar/2$ with 50% probability and $-\hbar/2$ with 50% probability.

Problem 5.- In the spinor χ given, find the constant A and calculate the probability that a measurement of S_X will yield $+\hbar/2$.

$$\chi = A\begin{pmatrix}3+4i\\12i\end{pmatrix}$$

Solution: First, we make sure that the wave function is normalized:

$$1 = \chi^+\chi = |A|^2(3-4i \quad -12i)\begin{pmatrix}3+4i\\12i\end{pmatrix} = |A|^2(9+16+144) = 169|A|^2$$

So, A = 1/13.

We understand that A can be any complex number with absolute value 1/13, but 1/13 is the simplest answer here.

To find the probability that a measurement of S_X will yield $+\hbar/2$ we write the spinor as a linear combination of eigenvectors of the operator S_X:

$$A\begin{pmatrix} 3+4i \\ 12i \end{pmatrix} = a\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} + b\begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix} \rightarrow a = \frac{A(3+16i)}{\sqrt{2}}$$

The probability is $|a|^2 = \left|\frac{A(3+16i)}{\sqrt{2}}\right|^2 = \frac{1}{169}\frac{1}{2}(9+256) = \frac{265}{338}$

Problem 6.-

a) Normalize the spinor:

$$\chi = A\begin{pmatrix} 1-2i \\ 2 \end{pmatrix}$$

b) Find the probabilities of measurements of S_z.
c) Find the probabilities of measurements of S_x.

Solution:

a) To normalize the spinor, we need to make sure that the sum of the squares of the absolute value of the coefficients is equal to one:

$$\chi = A\begin{pmatrix} 1-2i \\ 2 \end{pmatrix} \qquad |A(1-2i)|^2 + |2A|^2 = 1 \rightarrow |A|^2(1+4+4) = 1 \rightarrow |A|^2 = \frac{1}{9}$$

"A" can be a complex number, so in general it could be $\frac{1}{3}e^{i\alpha}$ for any real α, but we can choose to make A=$\frac{1}{3}$

b) Notice that the spinor is already written as a linear combination of the eigen functions of S_z:

$$\chi = A\begin{pmatrix} 1-2i \\ 2 \end{pmatrix} = A(1-2i)\begin{pmatrix} 1 \\ 0 \end{pmatrix} + 2A\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

The only possible results are the eigenvalues $\hbar/2$ and $-\hbar/2$. The probabilities are the coefficients squared:

Probability of $\hbar/2 = |A(1-2i)|^2 = \dfrac{1+4}{9} = 5/9$

Probability of $-\hbar/2 = |2A|^2 = \dfrac{4}{9} = 4/9$

The expectation value is calculated as usual, by multiplying the probabilities by the eigen values:

$$\langle S_Z \rangle = (\hbar/2)(5/9) + (-\hbar/2)(4/9) = \frac{\hbar}{18}$$

c) We need to write the spinor as a linear combination of eigen functions of S_x:

$$\chi = \begin{pmatrix} (1-2i)/3 \\ 2/3 \end{pmatrix} = X\begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} + Y\begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}$$

This gives us two simultaneous linear equations:

$$(1-2i)/3 = X/\sqrt{2} + Y/\sqrt{2}$$
$$2/3 = X/\sqrt{2} - Y/\sqrt{2}$$

We can add them to get X:

$$(1-2i)/3 + 2/3 = X\sqrt{2} \rightarrow X = \frac{3-2i}{3\sqrt{2}}$$

The square of the absolute value of X will be the probability of getting Sx=$\hbar/2$:

Probability of $\hbar/2 = \left|\dfrac{3-2i}{3\sqrt{2}}\right|^2 = \dfrac{9+4}{18} = 13/18$

The probability of getting the other eigenvalue will be 1-13/18 = 5/18.

The expectation value: $\langle S_X \rangle = (\hbar/2)(13/18) + (-\hbar/2)(5/18) = \dfrac{2\hbar}{9}$