Quantum Mechanics

Identical particles

Problem 1.- Find the expectation value $\left\langle (x_1 - x_2)^2 \right\rangle$ for two particles in the simple harmonic oscillator potential in states $|0\rangle$ and $|1\rangle$ in the cases

a) If the particles are distinguishable.
b) If they are identical bosons.
c) If they are identical fermions.

Solution:

Let's call $|n\rangle$ the eigen function of the harmonic oscillator in one dimension with energy eigen value $(n+1/2)\hbar\omega$. The quantum number n is a non-negative integer.

To calculate the various expectation values, we will need:

$x = \sqrt{\frac{\hbar}{2m\omega}}\left(a^+ + a^-\right)$ This operator is obtained by adding $a^+ + a^-$

To apply the operator, recall that:

$$a^+|n\rangle = \sqrt{n+1}|n+1\rangle$$
$$a^-|n\rangle = \sqrt{n}|n-1\rangle$$

The expectation value of the square of x can be found using this operator:

$$\left\langle x^2 \right\rangle = \langle n|x^2|n\rangle = \langle n|\left(\sqrt{\frac{\hbar}{2m\omega}}\left(a^+ + a^-\right)\right)^2|n\rangle = \frac{\hbar}{2m\omega}\langle n|\left(a^+ + a^-\right)^2|n\rangle$$

$$\left\langle x^2 \right\rangle = \frac{\hbar}{2m\omega}\langle n|\left(a^{+2} + a^- a^+ + a^+ a^- + a^{-2}\right)|n\rangle =$$
$$= \frac{\hbar}{2m\omega}\langle n|\left(\sqrt{(n+1)(n+2)}|n+2\rangle + (2n+1)|n\rangle + \sqrt{n(n-1)}|n-2\rangle\right)$$

Notice that only the term in the middle gives us a nonzero result, which is equal to:

$$\left\langle x^2 \right\rangle = \frac{\hbar}{2m\omega}(2n+1)$$

In our problem: $\langle 0|x^2|0\rangle = \frac{\hbar}{2m\omega}$ and $\langle 1|x^2|1\rangle = \frac{3\hbar}{2m\omega}$

We also know, just from symmetry that the expectation value of x vanishes for a harmonic oscillator.

To get the "cross integral" $\langle 0|x|1\rangle$ we again use the operator above:

$$\langle 0|x|1\rangle = \langle 0|\left(\sqrt{\frac{\hbar}{2m\omega}}\left(a^+ + a^-\right)\right)|1\rangle = \sqrt{\frac{\hbar}{2m\omega}}\langle 0|\left(a^+ + a^-\right)|1\rangle = \sqrt{\frac{\hbar}{2m\omega}}\langle 0|\left(\sqrt{2}|2\rangle + 1|0\rangle\right) = \sqrt{\frac{\hbar}{2m\omega}}$$

a) ***Distinguishable particles***:

$$\left\langle (x_1 - x_2)^2 \right\rangle = \left\langle x^2 \right\rangle_a + \left\langle x^2 \right\rangle_b - 2\langle x\rangle_a \langle x\rangle_b = \frac{\hbar}{2m\omega}(2(0)+1) + \frac{\hbar}{2m\omega}(2(1)+1) - 0 = \frac{2\hbar}{m\omega}$$

b) ***Identical Bosons***:

$$\left\langle (x_1 - x_2)^2 \right\rangle = \left\langle x^2 \right\rangle_a + \left\langle x^2 \right\rangle_b - 2\langle x\rangle_a \langle x\rangle_b - 2\left|\langle x\rangle_{ab}\right|^2 =$$

$$\frac{\hbar}{2m\omega}(2(0)+1) + \frac{\hbar}{2m\omega}(2(1)+1) - 0 - 2\frac{\hbar}{2m\omega} = \frac{\hbar}{m\omega}$$

b) ***Identical Fermions in the triplet state***:

$$\left\langle (x_1 - x_2)^2 \right\rangle = \left\langle x^2 \right\rangle_a + \left\langle x^2 \right\rangle_b - 2\langle x\rangle_a \langle x\rangle_b + 2\left|\langle x\rangle_{ab}\right|^2 =$$

$$\frac{\hbar}{2m\omega}(2(0)+1) + \frac{\hbar}{2m\omega}(2(1)+1) - 0 + 2\frac{\hbar}{2m\omega} = \frac{3\hbar}{m\omega}$$

Problem 2.- Consider that two particles interact through a potential that only depends on the vector $\vec{r} = \vec{r}_1 - \vec{r}_2$ and show that the Schrodinger equation can be separated in two equations, one for $\vec{r}$ and the other for the center of mass vector $\vec{R} = \dfrac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2}$

Solution: We write the kinetic energy in terms of the reduced mass, $\mu = \dfrac{m_1 m_2}{m_1 + m_2}$, the total mass and the vectors $\vec{R}$ and $\vec{r}$:

$$-\frac{\hbar^2}{2(m_1 + m_2)}\nabla_R^2\Psi - \frac{\hbar^2}{2\mu}\nabla_r^2\Psi + V(r)\Psi = E\Psi$$

Where $\Psi = \psi_R(R)\psi_r(r)$

$$-\frac{\hbar^2}{2(m_1+m_2)}\nabla_R^2\psi_R(R)\psi_r(r)-\frac{\hbar^2}{2\mu}\nabla_r^2\psi_R(R)\psi_r(r)+V(r)\psi_R(R)\psi_r(r)=E\psi_R(R)\psi_r(r)$$
$$-\frac{\hbar^2}{2(m_1+m_2)}\psi_r(r)\nabla_R^2\psi_R(R)-\frac{\hbar^2}{2\mu}\psi_R(R)\nabla_r^2\psi_r(r)+V(r)\psi_R(R)\psi_r(r)=E\psi_R(R)\psi_r(r)$$

Dividing by $\Psi=\psi_R(R)\psi_r(r)$:

$$-\frac{\hbar^2}{2(m_1+m_2)}\frac{\nabla_R^2\psi_R(R)}{\psi_R(R)}-\frac{\hbar^2}{2\mu}\frac{\nabla_r^2\psi_r(r)}{\psi_r(r)}+V(r)=E$$
$$-\frac{\hbar^2}{2(m_1+m_2)}\frac{\nabla_R^2\psi_R(R)}{\psi_R(R)}=E_R$$
$$-\frac{\hbar^2}{2\mu}\frac{\nabla_r^2\psi_r(r)}{\psi_r(r)}+V(r)=E_r$$

and $E=E_r+E_R$

The two separate equations are:

$$-\frac{\hbar^2}{2(m_1+m_2)}\nabla_R^2\psi_R(R)=E_R\psi_R(R)$$
$$-\frac{\hbar^2}{2\mu}\nabla_r^2\psi_r(r)+V(r)\psi_r(r)=E_r\psi_r(r)$$

and $E=E_r+E_R$