Quantum Mechanics

Perturbation theory

Problem 1.- Consider a perturbation to the simple harmonic oscillator given by:

$$H' = V(a^+ + a^-)^2$$

Which of the following gives the first-order shift in the n = 2 energy level due to the perturbation?

(A) 0 (B) V (C) $\sqrt{2}$ V (D) 2 $\sqrt{2}$ V (E) 5V

Solution: In first order perturbation theory:

$$E_n^1 = \langle n|H'|n\rangle$$

That is, the correction to the energy is given by the expectation value of the perturbing Hamiltonian with the unperturbed wave functions.
So:

$$E_n^1 = \langle n|V(a^+ + a^-)^2|n\rangle = V\langle n|(a^+ + a^-)^2|n\rangle = V\langle n|(a^{+^2} + a^+a^- + a^-a^+ + a^{-^2})|n\rangle$$

The operators squared will contribute zero to the energy since the resulting wave function will be n+2 or n-2, so:

$$E_n^1 = V\langle n|(a^+a^- + a^-a^+)|n\rangle$$

Now, using the rules: $a^+|n\rangle = \sqrt{n+1}|n+1\rangle$ and $a^-|n\rangle = \sqrt{n}|n-1\rangle$ we get:

$$E_n^1 = V\langle n|(a^+\sqrt{n}|n-1\rangle + a^-\sqrt{n+1}|n+1\rangle) = V\langle n|(n|n\rangle + (n+1)|n\rangle) = V(2n+1)$$

In the problem n=2, so $E_n^1 = 5V$

Answer: **E**

Problem 2.- A particle of mass m is confined by an infinite potential to 0<x<a. Use first order perturbation theory to calculate the correction to the eigen energies (ground state and excited states) for a perturbation H' given by:

$$H' = \lambda \sin\left(\frac{\pi x}{a}\right)$$

Solution: In first order perturbation theory:

$$E_n^1 = \langle n|H'|n\rangle$$

That is, the correction to the energy is given by the expectation value of the perturbing Hamiltonian with the unperturbed wave functions, so:

$$E_n^1 = \int_0^a \left(\sqrt{\frac{2}{a}}\sin\left(n\frac{\pi x}{a}\right)\right)^2 \lambda\sin\left(\frac{\pi x}{a}\right)dx$$

We notice that the integral vanishes for n>1, so the only correction is to n=1:

$$E_1^1 = \int_0^a \left(\sqrt{\frac{2}{a}}\sin\left(\frac{\pi x}{a}\right)\right)^2 \lambda\sin\left(\frac{\pi x}{a}\right)dx = \frac{2\lambda}{a}\int_0^a \sin^3\left(\frac{\pi x}{a}\right)dx$$

This can be rewritten as:

$$E_1^1 = -\frac{2\lambda}{a}\frac{a}{\pi}\int_0^a \sin^2\left(\frac{\pi x}{a}\right)d\cos\left(\frac{\pi x}{a}\right) = -\frac{2\lambda}{\pi}\int_1^{-1}(1-x^2)dx = \frac{2\lambda}{\pi}\left(x-\frac{x^3}{3}\right)_{-1}^{1} = \frac{8\lambda}{3\pi}$$