Quantum Mechanics

Probability current

Problem 1.- Consider $P_{ab} = \int_a^b \psi^* \psi dx$, the probability of a particle being in the region a<x<b .

Find an expression for the probability current, $\frac{dP_{ab}}{dt}$, which indicates the rate of change of the probability.

Solution:

We know that $P_{ab} = \int_a^b \psi^* \psi dx$, so:

$$\frac{dP_{ab}}{dt} = \frac{d}{dt}\int_a^b \psi^* \psi dx = \int_a^b \frac{\partial}{\partial t}\psi^* \psi dx = \int_a^b (\psi \frac{\partial}{\partial t}\psi^* + \psi^* \frac{\partial}{\partial t}\psi) dx$$

To simplify the last expression, we use Schrodinger's equation:

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 \psi}{\partial x^2} + V\psi \rightarrow \frac{\partial \psi}{\partial t} = i\frac{\hbar}{2m}\frac{\partial^2 \psi}{\partial x^2} - \frac{i}{\hbar}V\psi$$

and multiplying by the conjugate:

$$\psi^* \frac{\partial \psi}{\partial t} = i\frac{\hbar}{2m}\psi^* \frac{\partial^2 \psi}{\partial x^2} - \frac{i}{\hbar}V\psi^* \psi$$

Taking the conjugate of the above equation we get:

$$\psi \frac{\partial \psi^*}{\partial t} = -i\frac{\hbar}{2m}\psi \frac{\partial^2 \psi^*}{\partial x^2} + \frac{i}{\hbar}V\psi^* \psi$$

Summing the two equations together:

$$\psi \frac{\partial \psi^*}{\partial t} + \psi^* \frac{\partial \psi}{\partial t} = i\frac{\hbar}{2m}(\psi^* \frac{\partial^2 \psi}{\partial x^2} - \psi \frac{\partial^2 \psi^*}{\partial x^2})$$

Notice that we can also write this last expression as:

$$\psi \frac{\partial \psi^*}{\partial t} + \psi^* \frac{\partial \psi}{\partial t} = i\frac{\hbar}{2m}\frac{\partial}{\partial x}(\psi^* \frac{\partial \psi}{\partial x} - \psi \frac{\partial \psi^*}{\partial x})$$

Putting this equation back in the integral we get:

$$\frac{dP_{ab}}{dt} = \int_a^b i \frac{\hbar}{2m} \frac{\partial}{\partial x} (\psi^* \frac{\partial \psi}{\partial x} - \psi \frac{\partial \psi^*}{\partial x}) dx \, ,$$

which can be integrated by parts:

$$\frac{dP_{ab}}{dt} = i \frac{\hbar}{2m} (\psi^* \frac{\partial \psi}{\partial x} - \psi \frac{\partial \psi^*}{\partial x}) \Big|_a^b - \int_a^b i \frac{\hbar}{2m} (\psi^* \frac{\partial \psi}{\partial x} - \psi \frac{\partial \psi^*}{\partial x}) dx$$

Notice that the integral in the expression above is equal to zero because:

$$\int_a^b i \frac{\hbar}{2m} (\psi^* \frac{\partial \psi}{\partial x} - \psi \frac{\partial \psi^*}{\partial x}) dx = \int_a^b i \frac{\hbar}{2m} \frac{\partial}{\partial x} (\psi^* \psi - \psi \psi^*) dx = \int_a^b i \frac{\hbar}{2m} \frac{\partial}{\partial x} (0) dx = 0$$

This gives us the simpler expression:

$$\frac{dP_{ab}}{dt} = i \frac{\hbar}{2m} (\psi^* \frac{\partial \psi}{\partial x} - \psi \frac{\partial \psi^*}{\partial x}) \Big|_a^b = \text{J(a,t)-J(b,t)}$$

where: $J = i \frac{\hbar}{2m} (\psi \frac{\partial \psi^*}{\partial x} - \psi^* \frac{\partial \psi}{\partial x})$