Quantum Mechanics

Tunnel Effect

We have a particle of mass m that goes from left to right with constant linear momentum $p = \hbar k$ and kinetic energy $E = \frac{\hbar^2}{2m} k^2$. The wave function of such a particle is

$$\psi(x,t) = e^{i(kx-\omega t)} = \psi(x) e^{-i\omega t}$$

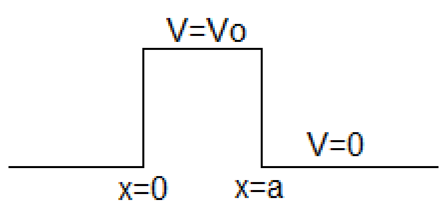
Where

$$\mathrm{k} = \frac{\sqrt{2\mathrm{mE}}}{\hbar}, \quad \omega = \frac{E}{\hbar} \text{ and } \psi(x) = e^{ik_1 x}$$

Notice that we cannot normalize this plane wave because it extends over all space.
To confirm that it has the correct linear momentum we apply the operator $\hat{p} = -i\hbar \frac{\partial}{\partial x}$, which gives us:

$$\hat{p}\psi(x) = -i\hbar \frac{\partial}{\partial x} e^{ikx} = k\hbar e^{ikx} = k\hbar\, \psi(x)$$

This confirms that $p = \hbar k$, which is a good quantum number.
Next, we consider the potential barrier shown below. The potential energy is zero everywhere except between x = 0, and x = a, where its value is V_0.



We will analyze what happens when this barrier gets in the way of the particle described above.

First case: Let us say that the total energy of the particle is greater than the potential V_0. Then the wave function can be written as three pieces:

$$\psi(\mathrm{x}) = \begin{cases} \mathrm{e}^{\mathrm{ikx}} + \mathrm{Be}^{-\mathrm{ikx}} & x < 0 \\ \mathrm{Ce}^{\mathrm{ik_2x}} + \mathrm{De}^{-\mathrm{ik_2x}} & 0 < x < a \\ \mathrm{Fe}^{\mathrm{ikx}} & x > a \end{cases}$$

Where

$$k_2 = \frac{\sqrt{2m(E - V_O)}}{\hbar}$$

We can interpret B as the amplitude of the reflected particle, C and D the amplitudes inside the barrier and F the amplitude of the transmitted wave after passing the barrier.

To find equations for the amplitudes, we notice that the functions must be continuous, so at the boundaries:

$$1 + B = C + D$$
$$Ce^{ik_2a} + De^{-ik_2a} = Fe^{ika}$$

Additionally, the first derivatives must be continuous too, so:

$$k - Bk = Ck_2 - Dk_2$$
$$Ck_2e^{ik_2a} - Dk_2e^{-ik_2a} = Fke^{ika}$$

Writing these four equations in matrix form, we get:

$$\begin{bmatrix} -1 & 1 & 1 & 0 \\ 0 & e^{ik_2a} & e^{-ik_2a} & -e^{ika} \\ k & k_2 & -k_2 & 0 \\ 0 & k_2e^{ik_2a} & -k_2e^{-ik_2a} & -ke^{ika} \end{bmatrix} \begin{bmatrix} B \\ C \\ D \\ F \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ k \\ 0 \end{bmatrix}$$

We solve the equations, obtaining:

$$B = \frac{(k^2 - k_2{}^2)\sin(k_2a)}{(k^2 + k_2{}^2)\sin(k_2a) + 2ikk_2\cos(k_2a)}$$

$$C = \frac{-2ke^{-ik_2a}(k + k_2)}{e^{ik_2a}(k_2 - k)^2 - e^{-ik_2a}(k_2 + k)^2}$$

$$D = \frac{-2k(k_2 - k)e^{ik_2a}}{e^{ik_2a}(k_2 - k)^2 - e^{-ik_2a}(k_2 + k)^2}$$

$$F = \frac{-4k_2ke^{-ika}}{e^{ik_2a}(k_2 - k)^2 - e^{-ik_2a}(k_2 + k)^2}$$

The transmission coefficient (T) is as follows:

$$T = |F|^2 = \frac{1}{1 + \frac{V_0{}^2}{4E(E - V_0)}\sin^2 k_2a}$$

Second case: If the total energy is less than V_0, the wave function will be:

$$\psi(x)=\begin{cases} e^{ikx}+Be^{-ikx} & x<0 \\ Ce^{k_2x}+De^{-k_2x} & 0<x<a \\ Fe^{ikx} & x>a \end{cases}$$

Where $\mathrm{k}_2=\dfrac{\sqrt{2\mathrm{m}(\mathrm{V}_0\text{-E})}}{\hbar}$

Like the first case, due to the continuity of the wave functions

$$1+\mathrm{B}=\mathrm{C}+\mathrm{D}$$
$$\mathrm{Ce}^{\mathrm{k}_2\mathrm{a}}+\mathrm{De}^{-\mathrm{k}_2\mathrm{a}}=\mathrm{Fe}^{\mathrm{ika}}$$

And continuity of the derivatives

$$\mathrm{ik}-\mathrm{Bik}=\mathrm{Ck}_2-\mathrm{Dk}_2$$
$$\mathrm{Ck}_2\mathrm{e}^{\mathrm{k}_2\mathrm{a}}-\mathrm{Dk}_2\mathrm{e}^{-\mathrm{k}_2\mathrm{a}}=\mathrm{Fike}^{\mathrm{ika}}$$

In matrix form:

$$\begin{bmatrix} -1 & 1 & 1 & 0 \\ 0 & e^{k_2a} & e^{-k_2a} & -e^{ika} \\ ik & k_2 & -k_2 & 0 \\ 0 & k_2e^{k_2a} & -k_2e^{-k_2a} & -ike^{ika} \end{bmatrix}\begin{bmatrix} B \\ C \\ D \\ F \end{bmatrix}=\begin{bmatrix} 1 \\ 0 \\ ik \\ 0 \end{bmatrix}$$

Solving the equations

$$B=\frac{(e^{-k_2a}-e^{k_2a})(k_2{}^2+k^2)}{(e^{k_2a}-e^{-k_2a})(k_2{}^2-k^2)-2ikk_2(e^{-k_2a}+e^{k_2a})}$$

$$C=\frac{2k(k-ik_2)e^{-k_2a}}{e^{-k_2a}(k-ik_2)^2+e^{k_2a}(k_2-ik)^2}$$

$$D=\frac{-2k(k+ik_2)e^{k_2a}}{e^{-k_2a}(k-ik_2)^2+e^{k_2a}(k_2-ik)^2}$$

$$F=\frac{-4ikk_2e^{-ika}}{e^{-k_2a}(k-ik_2)^2+e^{k_2a}(k_2-ik)^2}$$

In this case, the transmission coefficient is:

$$T = |F|^2 = \frac{1}{1 + \frac{V_0{}^2}{4E(V_0 - E)} \sinh^2 k_2 a}$$