Quantum Mechanics

Spin

Spin in quantum mechanics is different than in classical physics. We could understand it as angular momentum that is carried intrinsically, without the need for masses that rotate around an axis. This picture has been around for a long time because if you use a reasonable estimate of the size of the electron, its mass would have to move faster than the speed of light to produce its angular momentum.

Experimentally, the venerable Stern and Gerlach experiment is the crucial piece of evidence that showed the two possible states of spin (up and down), although it took a few years to be interpreted correctly.

Commutation equations for spin:

$$\left[S_x, S_y\right] = i\hbar S_z, \qquad \left[S_y, S_z\right] = i\hbar S_x, \qquad \left[S_z, S_x\right] = i\hbar S_y$$

The eigenvalues of S^2 are $s(s+1)\hbar^2$ and the ones of S_z are $m\hbar$, according to the equations:

$$S^2 \left| s, m \right\rangle = s(s+1)\hbar^2 \left| s, m \right\rangle$$
$$S_z \left| s, m \right\rangle = m\hbar \left| s, m \right\rangle$$

We can construct raising (S_+) and lowering (S_-) operators by linear combination:

$$S_\pm = S_x \pm i S_y$$

They have the effect of increasing or decreasing the quantum number *m*.

$$S_\pm \left| s, m \right\rangle = \hbar\sqrt{s(s+1) - m(m\pm 1)} \left| s, m \pm 1 \right\rangle$$

For spin ½ we have two eigenstates, which are called spin-up and spin down. Any general state of the spin (a spinor) can be expressed as a linear combination of these two states.

$$\chi = a\left|\uparrow\right\rangle + b\left|\downarrow\right\rangle = \begin{pmatrix} a \\ b \end{pmatrix} = a\begin{pmatrix} 1 \\ 0 \end{pmatrix} + b\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

The operators can be written with Pauli spin matrices:

$$S_x = \frac{\hbar}{2}\sigma_x = \frac{\hbar}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$S_y = \frac{\hbar}{2}\sigma_y = \frac{\hbar}{2}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

$$S_z = \frac{\hbar}{2}\sigma_z = \frac{\hbar}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$S_+ = \frac{\hbar}{2}(\sigma_x + i\sigma_y) = \hbar\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$S_- = \frac{\hbar}{2}(\sigma_x - i\sigma_y) = \hbar\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

$$S^2 = \frac{3\hbar}{4}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$